MA4266 Introduction to Algebraic Topology Notes

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Reference books:

(1). Munkres, J.R. (2000). *Topology*. Upper Saddle River, Nj: Prentice Hall, Inc.

(2). Adhikari, M. R. (2022). *Basic Topology 3: Algebraic Topology and Topology of Fiber Bundles*. Springer.

Contents

1. Fundamental Groups

1.1. *Homotopy and Path Homotopy*

Definition 1.1 (homotopy and nulhomotopy)**.** Let $I = [0,1]$ be the closed unit interval and X and Y be topological spaces. If f and f' are continuous maps from X to Y, then f is homotopic to f' if there exists a continuous map $F : X \times I \to Y$ such that

$$F(x,0) = f(x) \quad \text{and} \quad F(x,1) = f'(x) \quad \text{for all } x \in X.$$

The map F is a homotopy between f and f' and we write $f \simeq f'$. If

$$f \simeq f' \quad \text{where } f' \text{ is a constant map,} \quad \text{then} \quad f \text{ is nulhomotopic.}$$

In the above definition, think of a homotopy between f and f' as a continuous deformation from f to f'.

Example 1.1. Let A be a convex subspace of $\mathbb{R}^n$ and f and g be two continuous maps from X to A. Then, f and g are homotopic and the map

$$F(x,t) = (1-t)f(x) + tg(x) \quad \text{is a homotopy between them.}$$

Remark 1.1 (Chinese remainder theorem)**.** It is a well-known result in Ring theory that $\mathbb{Z}_{10} \cong \mathbb{Z}_2 \times \mathbb{Z}_5$, i.e. there exists an isomorphism between the rings $\mathbb{Z}_{10}$ and $\mathbb{Z}_2 \times \mathbb{Z}_5$ — this is merely an application of the Chinese remainder theorem. Actually, it would be *better* if one can construct an isomorphism between the two rings. In particular, we can define

$$\phi : \mathbb{Z}_{10} \to \mathbb{Z}_2 \times \mathbb{Z}_5 \quad \text{where} \quad \phi : [n]_{10} \mapsto ([n]_2, [n]_5).$$

Here, $[n]_k$ denotes the equivalence class of n modulo k.

Lemma 1.1. Suppose $f : I \to X$ is a path, then it is homotopic to some constant map.

In fact, the constant map can be chosen to be the map that maps I to x, where x is some point in $f([0,1])$.

Proof. Consider the map $F : I \times I \to X$ given by $F(s,t) = f((1-t)s)$. Then, F is continuous and $F(s,0) = f(s)$, $F(s,1) = f(0)$. Hence, F is a desired homotopy. □

Lemma 1.2. If X is path-connected, then for any paths $f, g \in X$, $f \simeq g$.

Note that if X is path-connected, then by definition, there exists a path in X from x_0 to x_1, i.e. there exists a continuous map $f : [0,1] \to X$ such that $f(0) = x_0$ and $f(1) = x_1$. We say that

$$x_0 \text{ is the initial point} \quad \text{and} \quad x_1 \text{ is the final point of the path } f.$$

Definition 1.2 (path homotopy). Two paths $f, f' : I \to X$ are path homotopic if

they have the same initial point x_0 and final point x_1

and if there exists a continuous map $F : I \times I \to X$ such that

$$F(s,0) = f(s) \text{ and } F(s,1) = f'(s)$$
$$F(0,t) = x_0 \text{ and } F(1,t) = x_1$$

for all $s,t \in I$. We say that F is a path homotopy between f and f' and we write $f \simeq_p f'$.

Definition 1.3 (homotopy equivalence class). If f is a path, we denote its path homotopy equivalence class by $[f]$.

So, $[f]$ is the set of all paths that are path homotopic to f. In other words, this set (or equivalence class) consists of all paths that can be continuously deformed into f while keeping the endpoints fixed. Also, to jump the gun, when we encounter the fundamental group in due course, we would be particularly interested in paths that are based at a specific point x_0 and share the same initial point and final point. These would be grouped into (no pun intended) path homotopy equivalence classes. For any topological space X, the fundamental group $\pi_1(X,x_0)$ would then be defined as

the set of equivalence classes under the operation of path concatenation.

Lemma 1.3. The relations $\simeq$ and $\simeq_p$ are equivalence relations.

Proof. We will prove the results concurrently.

- **Reflexivity:** Let f be a path. Then, it is homotopic to itself, i.e. $f \simeq f$. Note that $F(x,t) = f(x)$ is the desired homotopy. To see why, at $= 0$, the homotopy $F(x,0) = f(x)$ is exactly the original path; at time $t = 1$, the homotopy $F(x,1) = f(x)$ is still the original path.
- **Symmetry:** Suppose $f \simeq f'$, i.e. f is homotopic to f'. Let F be a homotopy between f to f'. Define $G(x,t) = F(x,1-t)$. Then, G is a homotopy between f' and f. To see why,

  $$G(x,0) = F(x,1) \text{ and } G(x,1) = F(x,0) \quad \text{which describe the situations at } t = 0 \text{ and } t = 1 \text{ respectively.}$$

  Hence, if F is a path homotopy, so is G.
- **Transitivity:** Suppose $f \simeq f'$ and $f' \simeq f''$. We wish to prove that $f \simeq f''$. Let

  $$F \text{ be a homotopy between } f \text{ and } f' \quad \text{and} \quad F' \text{ be a homotopy between } f' \text{ and } f''.$$

  Define $G : X \times I \to Y$ via

  $$G(x,t) = \begin{cases} F(x,2t) & \text{for } 0 \le x \le 1/2; \\ F'(x,2t-1) & \text{for } 1/2 \le x \le 1. \end{cases}$$

Then, G is well-defined, continuous by the pasting lemma, and is indeed a homotopy between f and f''. Lastly, if F and F' are path homotopies, then so is G. □

Lemma 1.4 (Munkres p. 330 Question 1). If

$$h, h' : X \to Y \text{ are homotopic} \quad \text{and} \quad k, k' : Y \to Z \text{ are homotopic},$$

then $k \circ h$ and $k' \circ h'$ are homotopic.

Proof. Let

$$F \text{ be a homotopy between } h \text{ and } h' \quad \text{and} \quad G \text{ be a homotopy between } k \text{ and } k'.$$

Then,

$$H(x,t) = G(F(x,t),t) \quad \text{is a homotopy between } k \circ h \text{ and } k' \circ h'.$$

To see why, we need to verify that

$$H(x,0) = (k \circ h)(x) \quad \text{and} \quad H(x,1) = (k' \circ h')(x) \quad \text{for all } x \in X.$$

This is trivial to see. □

Definition 1.4 (contractible space). A space X is contractible if the identity map $i_X : X \to X$ is nulhomotopic.

Lemma 1.5 (Munkres p. 330 Question 3). Every contractible space is path-connected.

Proof. Let X be a contractible space. Define

$$c_y : X \to X \quad \text{where} \quad c_y : x \mapsto y$$

be a constant map such that i_X is homotopic to, and F be the homotopy between i_X and c_y. It suffices to show that for every $x \in X$, there exists a path between x and y. Since F is a homotopy between i_X and c_y, then $F(x,0) = x$ and $F(x,1) = y$. Define $g : [0,1] \to X$ via $g(s) = F(x,s)$. Then, $g(0) = x$ and $g(1) = y$. So, g is the composition of F and $s \mapsto (x,s)$, which is continuous.

Hence, given $x, y \in X$, we can find a continuous path $g(s)$ connecting them, making X path-connected. □

Definition 1.5 (homotopy classes). Let X and Y be spaces. Let

$$[X,Y] \quad \text{denote the set of homotopy classes of maps of } X \text{ into } Y.$$

Geometrically, this is the set of different ways that the space X can be continuously mapped into the space Y, where two maps are said to be the *same* (or homotopic to be precise) if they can be continuously deformed into each other.

Lemma 1.6 (Munkres p. 366 Question 3)**.** Let $\mathscr{C}$ be a collection of spaces. Then,

$$\text{the relation of homotopy equivalence} \quad \text{is} \quad \text{an equivalence relation on } \mathscr{C}.$$

Proof. We need to check the three properties of an equivalence relation.

- **Reflexivity:** We need to show that a space $X \in \mathscr{C}$ is homotopy equivalent to itself, i.e. there exists a continuous map $f : X \to X$ such that f is homotopic to the identity map on X, denoted by id_X. To see why, consider the identity map $\mathrm{id}_X : X \to X$, for which $\mathrm{id}_X \circ \mathrm{id}_X = \mathrm{id}_X$, and id_X is trivially homotopic to itself via the constant homotopy.
- **Symmetry:** We need to show that

$$\text{if } X \in \mathscr{C} \text{ is homotopy equivalent to } Y \in \mathscr{C} \quad \text{then} \quad Y \text{ is homotopy equivalent to } X.$$

  Since X is homotopy equivalent to Y, then by definition, there exist continuous maps

$$f : X \to Y \text{ and } g : Y \to X \quad \text{such that} \quad g \circ f \simeq \mathrm{id}_X \text{ and } f \circ g \simeq \mathrm{id}_Y.$$

  Recall that $\simeq$ means homotopic. We showed that $\simeq$ is an equivalence relation in the proof of Lemma 1.3. Hence, $\simeq$ is symmetric, i.e. the roles of f and g can be reversed. We conclude that homotopy equivalence is symmetric.
- **Transitivity:** We need to show that

$$\text{if } X \in \mathscr{C} \text{ is homotopy equivalent to } Y \in \mathscr{C} \quad \text{and} \quad Y \in \mathscr{C} \text{ is homotopy equivalent to } Z \in \mathscr{C},$$

  then X is homotopy equivalent to Z. Since X is homotopy equivalent to Y, then by definition, there exist continuous maps

$$f : X \to Y \text{ and } g : Y \to X \quad \text{such that} \quad g \circ f \simeq \mathrm{id}_X \text{ and } f \circ g \simeq \mathrm{id}_Y.$$

  Similarly, since Y is homotopy equivalent to Z, then there exist continuous maps

$$h : Y \to Z \text{ and } k : Z \to Y \quad \text{such that} \quad k \circ h \simeq \mathrm{id}_Y \text{ and } h \circ k \simeq \mathrm{id}_Z.$$

  We need to construct maps $f' : X \to Z$ and $g' : Z \to X$, for which we use composition, i.e.

$$f' = h \circ f : X \to Z \quad \text{and} \quad g' = g \circ k : Z \to X.$$

  One checks that $g' \circ f' \simeq \mathrm{id}_X$ and $f' \circ g' \sim \mathrm{id}_Z$. The result follows. □

Proposition 1.1 (Munkres p. 330 Question 3)**.** Let X and Y be spaces. Then, the following hold:

(i) If Y is contractible, then for any X, $[X,Y]$ has a single element.

(ii) If X is contractible and Y is path-connected, then $[X,Y]$ has a single element.

Proof. We will only prove **(i)** as **(ii)** follows by joining several homotopies. To prove **(i)**, say $f : X \to Y$ and suppose i_Y is nulhomotopic to the map $c_y : Y \to Y$ given by $q \mapsto y$, where $y \in Y$ is a fixed element. It suffices to prove that f is homotopic to

$$\text{the constant map } c : X \to Y \quad \text{where} \quad c : p \mapsto y.$$

Let F be the homotopy between i_Y and c_y. Then, $F(q,0) = q$ and $F(q,1) = y$. Then, consider the map $G : X \times I \to Y$ given by $G(p,t) = F(f(p),t)$. Note that G is continuous, $G(p,0) = f(p)$ and $G(p,1) = y$. So, $G(p,t)$ is a homotopy between f and the constant map c. □

Example 1.2 (Munkres p. 330 Question 2)**.** Prove the following results:

(a) for any set X, the set $[X,I]$ has a single element;

(b) if Y is path-connected, then the set $[I,Y]$ has a single element

Solution.

(a) $[X,I]$ denotes the set of homotopy classes of maps of X into the unit interval $I = [0,1]$. It suffices to prove that I is contractible, i.e. we can continuously deform it to a singleton. We need to define a continuous family of maps (also known as a homotopy) that continuously shrinks every point $t \in I$ to 0 over time. In particular, this homotopy H can be written as

$$H(s,t) = (1-s)t \quad \text{which implies} \quad H(0,t) = t \text{ and } H(1,t) = 0.$$

We see that $H(0,t)$ is the identity map, and $H(1,t)$ is the constant map sending everything to 0.

(b) Similarly, $[I,Y]$ denotes the set of homotopy classes of maps of the unit interval I into a path-connected set Y. Our goal is to prove that any map $f : I \to Y$ is homotopic to a constant map. This can be done by considering the homotopy

$$F(s,t) = f((1-s)t) \quad \text{which implies} \quad F(0,t) = f(t) \text{ and } F(1,t) = f(0) \text{ is constant.}$$

So, f_1 is homotopic to the constant map equal to $f_1(0)$ and f_2 is homotopic to the constant map equal to $f_2(0)$. Since Y is path-connected, then

$$\text{any two points in } Y \text{ can be joined by a single path.}$$

So, there exists a path $\gamma : I \to Y$ such that $\gamma(0) = f_1(0)$ and $\gamma_1 = f_2(0)$. Let $F(s,t) = \gamma(s)$, which is a homotopy between the constant maps $t \mapsto f_1(0)$ and $t \mapsto f_2(0)$. Since $\simeq$ is an equivalence relation, i.e.

$$f_1 \sim (t \mapsto f_1(0)) \sim (t \mapsto f_2(0)) \sim f_2,$$

we obtain a homotopy between f_1 and f_2. □

Theorem 1.1 (Munkres p. 330 Question 3). I is contractible.

Proof. To prove that I is contractible, we need to construct a homotopy between the identity map on I and a constant map. Define the homotopy $H : I \times I \to I$ by the formula

$$H(x,t) = (1-t)x \quad \text{so} \quad H(x,0) = x \text{ and } H(x,1) = 0.$$

As such, $H(x,0)$ is the identity map on I and $H(x,1)$ is the constant map that sends every $x \in I$ to 0. Hence, H continuously deforms the identity map on I to a constant map. □

Theorem 1.2 (Munkres p. 330 Question 3). $\mathbb{R}$ is contractible.

Proof. Similar to our proof that I is constructible, we can consider the homotopy $H : I \times I \to I$ defined by $H(x,t) = (1-t)x$. □

1.2. *Fundamental Group*

Definition 1.6. Let f be a path in X from x_0 to x_1 and g be a path in X from x_1 to x_2. We define the product $f * g$ of f and g to be the path h given by the following equations:

$$h(s) = \begin{cases} f(2s) & \text{for } 0 \le s \le 1/2; \\ g(2s-1) & \text{for } 1/2 \le s \le 1. \end{cases}$$

$h = f * g$ is well-defined and continuous by the pasting lemma. Moreover, it is a path in X from x_0 to x_2. We can think of h as the path whose first half is the path f and whose second half is the path g.

Proposition 1.2. The product operation on paths induces a well-defined operation on path homotopy classes, which is still denoted by $*$. This is defined by the operation $[f] * [g] = [f * g]$.

Proof. We shall prove that the product operation on paths, denoted by $*$, is well-defined. That is, if

$$f \simeq_p f' \text{ and } g \simeq_p g' \quad \text{then} \quad f * g \simeq_p f' * g'.$$

Let

F be a path homotopy between f and f' and G be a path homotopy between g and g'.

Then, the map H, defined by

$$H(x,t) = \begin{cases} F(2x,t) & \text{if } 0 \le x \le 1/2, 0 \le t \le 1; \\ G(2x-1,t) & \text{if } 1/2 \le x \le 1, 0 \le t \le 1 \end{cases}$$

is a path homotopy between $f * g$ and $f' * g'$. □

Definition 1.7 (constant path). Let X be a topological space. Given $x \in X$, let e_x denote the constant path at x, i.e.

$$\text{the path } e_x : I \to X \quad \text{which carries all of } I \text{ to the point } x.$$

Then, $[e_x]$ is the path homotopy equivalence class of e_x.

Definition 1.8 (reverse path). Let f be a path in X from x_0 to x_1, then we define

$$\overline{f} \text{ to be the reverse of } f \quad \text{where} \quad \overline{f}(s) = f(1-s).$$

Proposition 1.3. The reverse operation on paths induces a well-defined operation (inverse) on path homotopy classes defined by the equation $[f]^{-1} = [\overline{f}]$.

Proof. Let $f \simeq_p g$. Then, it suffices to prove that $\overline{f} \simeq_p \overline{g}$. Let F be a path homotopy between f and g. Then, we define G via $G(x,t) = F(1-x,t)$, which is a path homotopy between f' and g'. □

Definition 1.9 (fundamental group). Let X be a topological space and $x_0 \in X$ be a point. We say that (X, x_0) is a pointed topological space.

- A path in X that begins and ends at x_0 is called a loop based at x_0.
- The set of path homotopy class of loops based at x_0 with the operation $*$ is called the fundamental group of X relative to the base point x_0, which is denoted by $\pi_1(X, x_0)$.

The elements of $\pi_1(X, x_0)$ are the path homotopy equivalence classes and the group operation is defined by $[f] * [g] = [f * g]$, the identity element is $[e_{x_0}]$ and the inverse of $[f]$ is $[\overline{f}]$.

Sometimes, $\pi_1(X, x_0)$ is called the first homotopy group of X, which term implies that there is a second homotopy group, and so on. There are indeed groups $\pi_1(X, x_0)$ for all positive integers n, but we shall not study them here. They are part of the general subject called Homotopy Theory.

Remark 1.2. The composite function $f \circ g$ involves applying g first, then f. Unlike function composition, for path homotopy equivalence classes, the concatenation $[f] * [g]$ implies that we apply $[f]$ first, then $[g]$.

Example 1.3 (fundamental group of $\mathbb{R}^n$ and convex subsets of $\mathbb{R}^n$). Let $\mathbb{R}^n$ denote the Euclidean n-space. Then, $\pi_1(\mathbb{R}^n, x_0)$ is the trivial group. More generally, more any convex subset of $\mathbb{R}^n$, $\pi_1(X, x_0)$ is the trivial group. In particular, the unit ball

$$B^n = \left\{x : x_1^2 + \ldots + x_n^2 \leq 1\right\} \quad \text{has trivial fundamental group.}$$

To see why, let f be a loop based at x_0, then the straight-line homotopy is a path homotopy between f and the constant path at x_0.

Since $\pi_1(X,x_0)$ depends on the path component of X containing x_0, it is usual to deal with only path-connected spaces when studying the fundamental group. Also, by constructing the path homotopy explicitly, we can derive the following theorem:

Theorem 1.3 (fundamental group is a group).

- **Associativity:** If f,g,h are paths for which $f(1)=g(0)$ and $g(1)=h(0)$, then

$$[f]*([g]*[h]) = ([f]*[g])*[h]$$

- **Right and left identities:** If f is a path in X from x_0 to x_1, then

$$[f]*[e_{x_1}] = [f] \quad \text{and} \quad [e_{x_0}]*[f] = [f].$$

- **Inverse:** If f is a path in X from x_0 to x_1, then

$$[f]*[\overline{f}] = [e_{x_0}] \quad \text{and} \quad [\overline{f}]*[f] = [e_{x_1}].$$

Definition 1.10 (conjugation map). Let α be a path in X from x_0 to x_1. We define the conjugation map

$$\widehat{\alpha} : \pi_1(X,x_0) \to \pi_1(X,x_1) \quad \text{via} \quad \widehat{\alpha}([f]) = [\overline{\alpha}]*[f]*[\alpha].$$

The map

$$\widehat{\alpha} : \pi_1(X,x_0) \to \pi_1(X,x_1) \quad \text{defined by} \quad \widehat{\alpha}(\gamma) = [\alpha]^{-1}*\gamma*[\alpha]$$

depends only on the path homotopy class of α and the map is well-defined.

Theorem 1.4. $\widehat{\alpha}$ is a group isomorphism.

Proof. Note that $\widehat{\alpha}$ is a group homomorphism. It is easy to construct the inverse of $\widehat{\alpha}$, which implies $\widehat{\alpha}$ is invertible, and hence, a bijective map. □

Example 1.4 (Munkres p. 334 Question 2). Let α be a path in X from x_0 to x_1; let β be a path in X from x_1 to x_2. Show that if

$$\gamma = \alpha*\beta \quad \text{then} \quad \widehat{\gamma} = \widehat{\beta}\circ\widehat{\alpha}.$$

Solution. Since α is a path in X from x_0 to x_1 and β is a path in X from x_1 to x_2, then $\gamma = \alpha*\beta$ is a path in X from x_0 to x_2. The conjugation map $\widehat{\gamma}$ is defined as follows:

$$\widehat{\gamma} : \pi_1(X,x_0) \to \pi_1(X,x_2) \quad \text{via} \quad \widehat{\gamma}([f]) = [\overline{\gamma}]*[f]*[\gamma]$$

Here, f is a loop based at x_0 and $[\bar{\gamma}]$ is the homotopy class of the reverse of the path γ. Let g and h be loops based at x_1 and x_0 respectively. Writing γ as $\alpha * \beta$, we have

$$\begin{aligned}
\widehat{\gamma}([f]) &= [\bar{\gamma}] * [f] * [\gamma] \text{ by definition} \\
&= [\overline{\alpha * \beta}] * [f] * [\alpha * \beta] \text{ by writing } \gamma = \alpha * \beta \\
&= \left([\overline{\beta}] * [\overline{\alpha}]\right) * [f] * [\alpha * \beta] \text{ since } \overline{\alpha * \beta} = [\overline{\beta}] * [\overline{\alpha}] \\
&= [\overline{\beta}] * ([\overline{\alpha}] * [f] * [\alpha]) * [\beta] \text{ by associativity} \\
&= [\overline{\beta}] * [\overline{\alpha}] * [h] * [\alpha] * [\beta] \\
&= \widehat{\beta}\left(\widehat{\alpha}(f)\right)
\end{aligned}$$

The result follows. □

Corollary 1.1. If X is path-connected and x_0, x_1 are two points of X, then

$$\pi_1(X, x_0) \cong \pi_1(X, x_1).$$

In particular, if X is path-connected, for any $x \in X$, all the groups in $\pi_1(X, x)$ are isomorphic.

Proposition 1.4. Let x_0, x_1 be points of a path-connected space X. Then,

$\pi_1(X, x_0)$ is Abelian if and only if for all paths from x_0 to x_1 denoted by α, β, we have $\widehat{\alpha} = \widehat{\beta}$.

Proof. We first prove the forward direction. Suppose $\pi_1(X, x_0)$ is Abelian. Then, for any paths α and β, note that $\alpha * \overline{\beta}$ is a loop at x_0. So,

$$[\beta * \overline{\alpha}] * [\alpha * \overline{\beta}] = e \quad \text{which shows that } \widehat{\alpha} \text{ and } \widehat{\beta} \text{ are inverses of each other.}$$

The reverse direction is easy to see, i.e. need to prove $[f] * [g] = [g] * [f]$. □

Definition 1.11 (simply-connected)**.** A space X is

simply-connected if and only if X is a path-connected space and $\pi_1(X, x_0)$ is trivial for all $x \in X$.

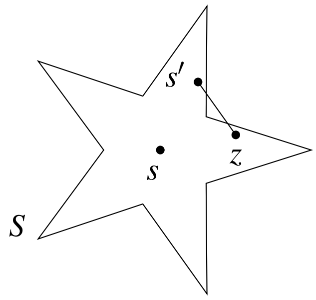
Example 1.5. Any convex subset of $\mathbb{R}^n$ is simply-connected. A connected open subset U of $\mathbb{C} = \mathbb{R}^2$ is simply-connected if and only if its complement $\overline{\mathbb{C}} \setminus U$ in $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ is connected. This requires the Riemann mapping theorem. Here, $\overline{\mathbb{C}}$ denotes the set of extended complex numbers.

Definition 1.12 (star convex)**.** A subset $A \subseteq \mathbb{R}^n$ is star convex if

for some $a_0 \in A$, all the line segments joining a_0 to other points of A lie in A.

Example 1.6 (Munkres p. 334 Question 1)**.** Find a star convex set that is not convex.

Solution. We shall consider a star S (*literally*). Consider the centre of the star $s \in S$. Then, for any other point $z \in S$, all the line segments joining s to z lie in S.



However, S is not a convex as there exists $s' \in S$ such that the line segment from s' to z lies in S. $\square$

Lemma 1.7 (Munkres p. 334 Question 334)**.** If A is a star convex set, then A is simply-connected.

Proof. Let $p(s)$ be a loop in A, where $0 \leq s \leq 1$. Then, we can construct a straight-line homotopy

$$H(s,t) = ta + (1-t)p(s) \quad \text{with} \quad H(s,0) = p(s) \text{ and } H(s,1) = a.$$

This implies p is homotopic to a point, so A is simply-connected. $\square$

Lemma 1.8. Let X be a simply-connected space. Then, any two paths having the same initial and final points are path homotopic.

Proof. Let α and β be two paths from x_0 to x_1. Then, $\alpha * \overline{\beta}$ is defined and is a loop on X based at x_0. Since X is simply-connected, this loop is path homotopic to the constant loop at x_0. Thus implies

$$[\alpha * \overline{\beta}] * [\beta] = [e_{x_0}] * [\beta]$$

for which it follows that $[\alpha] = [\beta]$. $\square$

Example 1.7 (Munkres p. 335 Question 3)**.** Let x_0 and x_1 be points of the path-connected space X. Show that

$$\pi_1(X,x_0) \text{ is Abelian} \quad \text{if and only if} \quad \text{for every pair } \alpha,\beta \text{ of paths from } x_0 \text{ to } x_1, \text{ we have } \widehat{\alpha} = \widehat{\beta}.$$

Solution. Suppose $\pi_1(X,x_0)$ is Abelian. Then, for any loops $[f],[g]$ based at x_0, we have $[f]*[g] = [g]*[f]$. Say α and β are paths from x_0 to x_1. Then, $[\alpha * \overline{\beta}]$ is the homotopy class of loops based at x_0. Recall that by Definition 1.10, it suffices to prove that

$$[\overline{\alpha}] * [f] * [\alpha] = [\overline{\beta}] * [f] * [\beta].$$

Replacing g with $\alpha * \overline{\beta}$, we have

$$[f] * [\alpha] * [\overline{\beta}] = [\alpha] * [\overline{\beta}] * [f].$$

I believe that the result follows obviously from here. The reverse direction can be proven similarly. $\square$

1.3. *Functorial Property*

Lemma 1.9. If $k : X \to Y$ is a continuous map and

$$F \text{ is a path homotopy in } X \quad \text{between } f \text{ and } f',$$

then

$$k \circ F \text{ is a path homotopy in } X \quad \text{between } k \circ f \text{ and } k \circ f'.$$

Lemma 1.10. Let $k : X \to Y$ be a continuous map and f and g be paths in X with $f(1) = g(0)$. Then,

$$k \circ (f * g) = (k \circ f) * (k \circ g).$$

Definition 1.13 (induced homomorphism)**.** Suppose $h : X \to Y$ is a continuous map that carries $x_0 \in X$ to $y_0 \in Y$. We denote this by writing $h : (X, x_0) \to (Y, y_0)$. Then, define the induced homomorphism on fundamental groups, h_*, to be

$$h_* : \pi_1(X, x_0) \to \pi_1(Y, y_0) \quad \text{by} \quad h_*([f]) = [h \circ f].$$

Note that $h_*(\gamma)$ is well-defined, i.e. if F is a path homotopy between the paths f and f', then $h \circ F$ is a path homotopy between the paths $h \circ f = h \circ f'$.

Remark 1.3. The induced homomorphism h_* depends not only on the map $h : X \to Y$ but also on the choice of the base point x_0. To see why, if x_0 and x_1 are two different points of X, we cannot use the same symbol h_* to denote two different homomorphisms, with one having domain $\pi_1(X, x_0)$ and the other having domain $\pi_1(X, x_1)$. Even if X is path-connected (which implies the groups are isomorphic), they are still not the same group. In such a case, we use the notation

$$\begin{aligned} (h_{x_0})_* : \pi_1(X, x_0) \to \pi_1(Y, y_0) \quad & \text{for the first homomorphism} \quad \text{and} \\ (h_{x_1})_* \quad & \text{for the second (defined similarly)} \end{aligned}$$

Lemma 1.11. $h_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ is a group homomorphism.

Theorem 1.5 (functiorial property of π_1)**.** The following hold:

(i) If $h : (X, x_0) \to (Y, y_0)$ and $k : (Y, y_0) \to (Z, z_0)$ are continuous, then $(k \circ h)_* = k_* \circ h_*$.

(ii) If $i : (X, x_0) \to (X, x_0)$ is the identity map, then i_* is the identity homomorphism.

Proof. **(ii)** is easy to prove — we have $i_*([f]) = [i \circ f] = [f]$. To prove **(i)**, we have

$$(k \circ h)_*([f]) = [(k \circ h) \circ f]$$
$$(k_* \circ h_*)([f]) = k_*(h_*([f])) = k_*([h \circ f]) = [k \circ (h \circ f)]$$

and these two expressions are equivalent by associativity. □

Corollary 1.2. If $h : (X, x_0) \to (Y, y_0)$ is a homeomorphism of X and with Y, then h_* is an isomorphism of $\pi_1(X, x_0)$ with $\pi_1(Y, y_0)$.

Proof. Let

$$k : (Y, y_0) \to X, x_0) \text{ be the inverse of } h \quad \text{where} \quad h : (X, x_0) \to (Y, y_0).$$

Then, $k_* \circ h_* = (k \circ h)_* = i_*$, where i is the identity map of (X, x_0). Similarly, we have $h_* \circ k_* = j_*$, where j is the identity map of (Y, y_0). Since i_* and j_* are the identity homomorphism of their respective groups, then k_* is the inverse of h_*. □

Theorem 1.6. Let $h : X \to Y$ be continuous with $h(x_0) = y_0$ and $h(x_1) = y_1$. Also, let α be a path in X from x_0 to x_1. Define $\beta = h \circ \alpha$. Then,

$$\widehat{\beta} \circ (h_{x_0})_* = (h_{x_1})_* \circ \widehat{\alpha}, \text{ i.e. the following diagram commutes:}$$

$$\begin{array}{ccc} \pi_1(X, x_0) & \xrightarrow{(h_{x_0})_*} & \pi_1(Y, y_0) \\ \widehat{\alpha} \downarrow & & \downarrow \widehat{\beta} \\ \pi_1(X, x_1) & \xrightarrow{(h_{x_1})_*} & \pi_1(Y, y_1) \end{array}$$

Example 1.8 (Munkres p. 335 Question 7)**.** Let G be a topological group $\cdot$ and identity element x_0. Let

$$\Omega(G, x_0) \quad \text{denote the set of all loops in } G \text{ based at } x_0.$$

If $f, g \in \Omega(G, x_0)$, then we define a loop $f \otimes g$ by the formula $(f \otimes g)(s) = f(s) \cdot g(s)$.

(a) Prove that this operation makes the set $\Omega(G, x_0)$ into a group.

(b) Prove that this operation induces a group operation $\otimes$ on $\pi_1(G, x_0)$.

(c) By computing

$$(f * e_{x_0}) \otimes (e_{x_0} * g),$$

prove that the two group operations $*$ and $\otimes$ on $\pi_1(G, x_0)$ are the same.

(d) Prove that $\pi_1(G, x_0)$ is Abelian.

Solution.

(a) We first prove that Ω contains the identity element, i.e. there exists $e_{x_0} \in \Omega(G, x_0)$ such that

$$f \otimes e_{x_0} = f = e_{x_0} \otimes f \quad \text{for all } f \in \Omega(G, x_0).$$

We shall define the constant loop $e_{x_0} : I \to G$ such that $e_{x_0}(s) = x_0$ for all $s \in I$. Then,

$$(f \otimes e_{x_0})(s) = f(s) \cdot e_{x_0}(s) = f(s) \cdot x_0 = f(s)$$

and the proof that $e_{x_0} \otimes f = f$ is similar.

We then prove that Ω has an inverse element. That is,

$$\text{for every } f \in \Omega(G, x_0), \quad \text{there exists } f^{-1} \in \Omega(G, x_0) \quad \text{such that} \quad f \otimes f^{-1} = e_{x_0}.$$

Define $f^{-1}(s) = (f(s))^{-1}$, where the inverse operation on the RHS is with respect to the group G. Hence,

$$\left(f \otimes f^{-1}\right)(s) = f(s) \cdot f^{-1}(s) = f(s) \cdot (f(s))^{-1} = x_0$$

so it follows that $f \otimes f^{-1} = e_{x_0}$.

Lastly, we prove that Ω is associative under $\otimes$, i.e.

$$((f \otimes g) \otimes h)(s) = (f \otimes (g \otimes h))(s).$$

This is in fact equivalent to

$$(f(s) \cdot g(s)) \cdot h(s) = f(s) \cdot (g(s) \cdot h(s)) \quad \text{which is true since } G \text{ is associative under } \cdot.$$

(b) By definition, the fundamental group $\pi_1(G, x_0)$ denotes the set of homotopy classes of loops based at x_0. We have $[f] \otimes [g] = [f \otimes g]$, so $\otimes$ is a well-defined operation. We need to prove that

$$f \sim f' \text{ and } g \sim g' \quad \text{implies} \quad f \otimes g \sim f' \otimes g'.$$

Let $H_f : I \times I \to G$ be a homotopy between f and f' and $H_g : I \times I \to G$ be a homotopy between g and g'. Define

$$H(s,t) = H_f(s,t) \cdot H_g(s,t).$$

Then,

$$H(s,0) = H_f(s,0) \cdot H_g(s,0) = f(s) \cdot g(s) \quad \text{and} \quad H(s,1) = H_f(s,1) \cdot H_g(s,1) = f'(s) \cdot g'(s).$$

So, H is indeed a homotopy between $f \otimes g$ and $f' \otimes g'$. Combining the result in **(a)**, **(b)** follows.

(c) We have

$$((f * e_{x_0}) \otimes (e_{x_0} * g))(s) = (f * e_{x_0})(s) \cdot (e_{x_0} * g)(s)$$

Recall that

$$(f * e_{x_0})(s) = \begin{cases} f(2s) & \text{if } 0 \leq s \leq 1/2; \\ x_0 & \text{if } 1/2 \leq s \leq 1 \end{cases} \quad \text{and} \quad (e_{x_0} * g)(s) = \begin{cases} x_0 & \text{if } 0 \leq s \leq 1/2; \\ g(2s-1) & \text{if } 1/2 \leq s \leq 1. \end{cases}$$

By the pasting lemma,

$$(f * e_{x_0})(s) \cdot (e_{x_0} * g)(s) = \begin{cases} f(2s) \cdot x_0 = f(2s) & \text{if } 0 \leq s \leq 1/2; \\ x_0 \cdot g(2s-1) = g(2s-1) & \text{if } 1/2 \leq s \leq 1. \end{cases}$$

This is precisely the definition of $f * g$, so the result follows.

(d) Consider $[f], [g] \in \pi_1(G, x_0)$. Then,

$$\begin{aligned} [f] * [g] &= [f] \otimes [g] \quad \text{by } \mathbf{(c)} \\ &= [e_{x_0} * f] \otimes [g * e_{x_0}] \\ &= [g * f] \end{aligned}$$

which is equivalent to $[g] * [f]$, implying that the fundamental group is Abelian. □

1.4. *Covering Spaces*

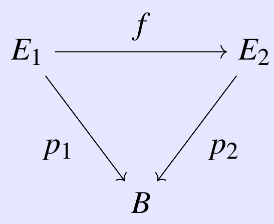
Definition 1.14 (B-space)**.** A B-space or a space over a topological space of interest B consists of

a topological space E and a continuous map $p : E \to B$ (known as a structural map).

Suppose

$$p_1 : E_1 \to B \text{ and } p_2 : E_2 \to B \quad \text{are } B\text{-spaces.}$$

Definition 1.15 (B-space map)**.** A B-space map from E_1 to E_2 is a continuous map $f : E_1 \to E_2$ such that $p_2 \circ f = p_1$, i.e. the following diagram commutes:



Furthermore, a B-space map $f : E_1 \to E_2$ is an isomorphism of B-spaces if there exists a B-space map $g : E_2 \to E_1$ such that

$$g \circ f = \mathrm{id}_{E_1} \quad \text{and} \quad f \circ g = \mathrm{id}_{E_2}.$$

Example 1.9 (projection map). For any topological space F, the projection map $p : B \times F \to B$, where $(b,a) \mapsto b$ makes $B \times F$ into a B-space (by convention, the projection map is denoted by π instead of p).

Example 1.10 (restriction map). Let $p : E \to B$ be a B-space. For any $B' \subseteq B$, define $E' = p^{-1}(B') \subseteq E$ and the restriction map $p|_{E'} : E' \to B'$. Then, E' becomes a space over B', called the restriction of p over B'.

Example 1.11 (identity map). If $p : E \to B$ is a B-space, the identity map $\mathrm{id}_E : E \to E$ is a B-space map.

Example 1.12. Let E_1 and E_2 be topological spaces. If

$$f : E_1 \to E_2 \text{ and } g : E_2 \to E_3 \quad \text{are } B\text{-spaces maps,}$$

the composite map $g \circ f : E_1 \to E_3$ is a B-space map. This is obvious since E_1 and E_2 are topological spaces and the maps f and g are continuous (since they are B-spaces maps).

Definition 1.16 (even cover). Let $p : E \to B$ be a B-space. An open set $U \subseteq B$ is evenly covered by p if there exists a non-empty discrete space F such that

$$p|_U \cong U \times F \quad \text{as } U\text{-spaces.}$$

That is to say,

$$p^{-1}(U) \subseteq E \quad \text{can be written as a non-empty disjoint union of open sets in } E, \text{ i.e. } \bigsqcup_{\alpha \in F} V_a.$$

such that $p|_{V_\alpha}$ is a homomorphism of V_α onto U. We say that the collection $\{V_\alpha\}$ is a partition of $p^{-1}(U)$ into slices.

Remark 1.4. If U is evenly covered by p and $W \subseteq U$ is open, then W is evenly covered by p.

Definition 1.17 (covering map). A covering space of B is a B-space $p : E \to B$ which is locally trivial (p is a covering map), i.e. for all $b \in B$, there exists an open neighbourhood U of b that is evenly covered by p. If this is the case,

$$p \text{ is a covering map} \quad \text{and} \quad E \text{ is a covering space of } B.$$

Remark 1.5. If $p : E \to B$ is a covering map, then the following hold:

(i) for all $b \in B$, the fiber $p^{-1}(b) \subseteq E$ given by the subspace topology from E has the discrete topology. To see why, based on the definition of a covering map , each point $e = p^{-1}(b)$ lies in some open set of E that is mapped homeomorphically onto some neighbourhood of B. So, each point in the fiber can be separated using disjoint open sets, giving the fiber the discrete topology.

(ii) $p : E \to B$ is an open map and surjective

(iii) p is a local homomorphism of E with B, i.e. for all $e \in E$, there exists an open neighbourhood V of e such that $p|_V : V \to p(V)$ is a homeomorphism

Example 1.13. The identity map $\mathrm{id}_B : B \to B$ is a covering map.

Example 1.14 (Munkres p. 341 Question 1)**.** For any non-empty discrete space F, the projection map $p : B \times F \to B$ such that $(b,a) \mapsto b$ is a covering map. To see why, note that p is continuous and surjective, and

$$p^{-1}(B) = \bigcup_{f \in F} B \times \{f\},$$

where each slice $B \times \{f\}$ is open in $B \times F$ (as F is discrete) and homeomorphic to B.

Example 1.15. If

$$p : E \to B \text{ and } p' : E' \to E \quad \text{are finite covering maps,}$$

then $p \circ p' : E' \to B$ is a finite covering map.

Theorem 1.7. The exponential map $p : \mathbb{R} \to \mathbb{S}^1 \subseteq \mathbb{C} \setminus \{0\}$ given by the equation

$$p(x) = (\cos 2\pi x, \sin 2\pi x) \text{ or equivalently } p(x) = \exp(2\pi i x) \quad \text{is a covering map.}$$

Proof. Note that the map p is continuous and surjective. We now prove that p is locally trivial. Consider some point $y_0 = \exp(2\pi x_0) \in \mathbb{S}^1$. We can choose an open interval $I \subseteq \mathbb{R}$ around y_0 such that $p|_I$ is injective[†].

Now, consider $p^{-1}(V)$, where $V = p(I) \subseteq \mathbb{S}^1$. Note that $p(x+1) = p(x)$ since $\cos 2\pi x$ and $\sin 2\pi x$ are periodic with period 1. Hence, $p^{-1}(V)$ consists of a disjoint union of intervals of the form $I_k = I + k$ for all $k \in \mathbb{Z}$. Each I_k is mapped homeomorphically onto V by the map p, so we conclude that p is locally trivial. □

Example 1.16. The map $p : \mathbb{R}^+ \to \mathbb{S}^1$ given by $x \mapsto \exp(2\pi i x)$ is surjective and a local homeomorphism, but it is not a covering map. This is because the point $\mathbb{S}^1 \ni b = (1,0)$ has no neighbourhood U

[†]Here is a YouTube video which shows how one can visualise a covering space of $\mathbb{S}^1$ via the exponential map $p : \mathbb{R} \to \mathbb{S}^1$

that is not evenly covered by p. To see why, the boundary of $\mathbb{R}^+$ at $x = 0$ disrupts the ability to create disjoint preimages that evenly cover the neighbourhoods of $\mathbb{S}^1$.

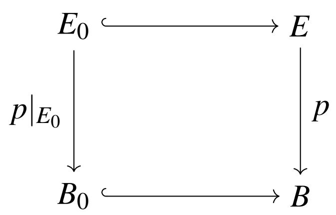
Example 1.17 (Munkres p. 341 Question 5). The map $p : \mathbb{S}^1 \to \mathbb{S}^1$ given by $z \mapsto z^2$ is a covering map. More generally, for any $n \in \mathbb{Z} \setminus \{0\}$, the map $p : \mathbb{S}^1 \to \mathbb{S}^1$ given by $z \mapsto z^n$ is a covering map.

To see why, for the general case, suppose $p(z) = w$. Consider $w \in \mathbb{C} \setminus \{0\}$, which has a small neighbourhood U around $\mathbb{C} \setminus \{0\}$ which avoids the branch cut of the complex logarithm. So, the pre-image $p^{-1}(w)$ consists of n distinct points, i.e.

$$z_k = |w|^{1/n} \exp\left(\frac{2k\pi}{n} i\right) \quad \text{such that} \quad p(z_k) = w.$$

Proposition 1.5. Let $p : E \to B$ be a covering map. If B_0 is a subspace of B and $E_0 = p^{-1}(B_0)$, then the map $p_0 : E_0 \to B_0$ by restricting p is a covering map.

Proof. Consider the following commutative diagram:



Given $b_0 \in B_0$, let

$$U \text{ be an open set in } B \text{ containing } b_0 \quad \text{that is evenly covered by } p.$$

Also, let $\{V_\alpha\}$ be a partition of $p^{-1}(U)$ into slices. So, $U \cap B_0$ is a neighbourhood of b_0 in B_0, and the sets $V_\alpha \cap E_0$ are disjoint open sets in E_0 such that

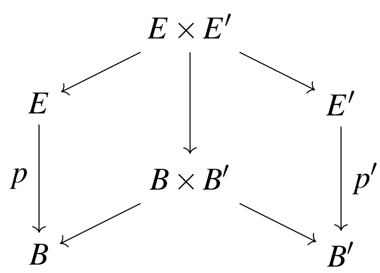
$$p^{-1}(U \cap B_0) = \bigcup_\alpha V_\alpha \cap E_0,$$

and $p : V_\alpha \cap E_0 \to U \cap B_0$ is a homeomorphism. □

Proposition 1.6. If $p : E \to B$ and $p' : E' \to B'$ are covering maps, then

$$p \times p' : E \times E' \to B \times B' \quad \text{is a covering map.}$$

Proof. Consider the following commutative diagram:



Given $b \in B$ and $b' \in B'$, let U and U' be neighbourhoods of b and b' respectively which are evenly covered by p and p'. Let $\{V_\alpha\}$ and $\{V'_\alpha\}$ be partitions of $p^{-1}(U)$ and $(p')^{-1}(U')$ into slices. Then, the inverse image under $p \times p'$ of the open set $U \times U'$ is the union of all the sets $V_\alpha \times V'_\alpha$. These are disjoint open sets of $E \times E'$, and each is mapped homeomorphically onto $U \times U'$ by $p \times p'$. □

Proposition 1.7. Let

$$q : X \to Y \text{ and } r : Y \to Z \quad \text{be covering maps.}$$

Define $p = r \circ q$. If $r^{-1}(z)$ is finite for all $z \in Z$, then p is a covering map.

Example 1.18 (covering map of torus). Let $\mathbb{T} = \mathbb{S}^1 \times \mathbb{S}^1$ denote the torus. The product map

$$p \times p : \mathbb{R} \times \mathbb{R} \to \mathbb{S}^1 \times \mathbb{S}^1 \subseteq \mathbb{C}^\times \times \mathbb{C}^\times \quad \text{is a covering of the torus} \quad \text{by the plane } \mathbb{R}^2.$$

Here, $\mathbb{C}^\times = \mathbb{C} \setminus \{\mathbf{0}\}$. Also, p maps $(x,y) \in \mathbb{R} \times \mathbb{R}$ to $\left(e^{2\pi ix}, e^{2\pi iy}\right) \in \mathbb{S}^1 \times \mathbb{S}^1$. Geometrically, think of having an infinite grid in $\mathbb{R}^2$ and first wrap it with respect to the vertical axis to obtain a hollow cylinder. Thereafter, wrap it with respect to the horizontal axis to obtain the torus $\mathbb{T}$.

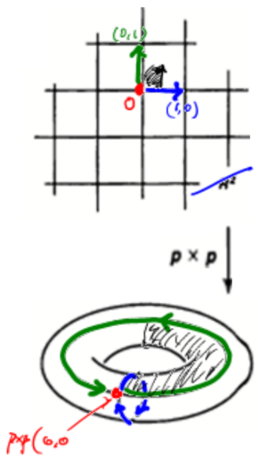


Figure 1: Covering space of the torus $\mathbb{T}$

Example 1.19 (figure-eight space). Let b_0 denote the point $p(0) \in \mathbb{S}^1$ and B_0 denote the figure-eight space, i.e.

$$B_0 = \left(\mathbb{S}^1 \times \{b_0\}\right) \cup \left(\{b_0\} \times \mathbb{S}^1\right) \quad \text{which is the union of } \mathbb{S}^1 \text{ and } \mathbb{S}^1 \quad \text{that have a point in common.}$$

This is an example of a space that is not simply-connected; it is the wedge sum of two circles, so B_0 does not contain any contractible looks that can be continuously shrunk to a point. The space $E_0 = p^{-1}(B)$ is the infinite grid

$$E_0 = (\mathbb{R} \times \mathbb{Z}) \cup (\mathbb{Z} \times \mathbb{R}) \quad \text{for which the grids intersect at } \mathbb{Z} \times \mathbb{Z}.$$

The map $p_0 : E_0 \to B_0$ obtained by restricting $p \times p$ is a covering map.

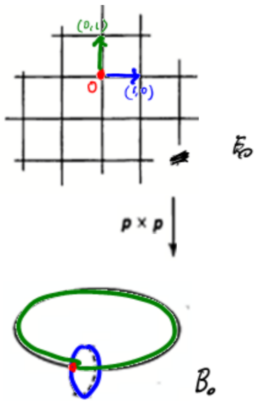


Figure 2: Covering space of the figure-eight space B_0

Example 1.20 (punctured plane)**.** Consider the covering map of the punctured plane by the open upper-half plane $\mathscr{H} = \mathbb{R} \times \mathbb{R}^+$ as follows:

$$\mathbb{R} \times \mathbb{R}^+ \to \mathbb{S}^1 \times \mathbb{R}^+ \to \mathbb{R}^2 \setminus \{\mathbf{0}\} = \mathbb{C}^\times \quad \text{given by} \quad (x,y) \mapsto \left(e^{2\pi i x}, r\right) \mapsto re^{2\pi i x}.$$

For the map $(x,y) \mapsto \left(e^{2\pi i x}, r\right)$, we see that the first coordinate $x \in \mathbb{R}$ is mapped onto the unit circle $\mathbb{S}^1$ via the complex exponential function $e^{2\pi i x}$, which wraps the real line onto the circle. $r = \sqrt{x^2 + y^2}$ measures the distance this point makes with the origin.

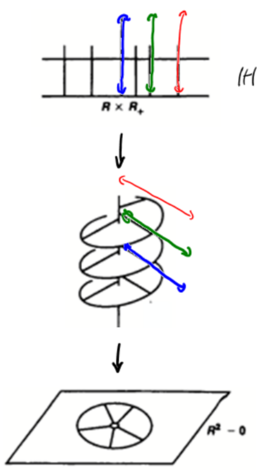


Figure 3: Covering space of the punctured plane

Proposition 1.8. Let $p : E \to B$ be a covering map. Suppose U is an open set of B that is evenly covered by p. If U is connected, then the partition of $p^{-1}(U)$ into slices is unique.

Proof. Suppose $\{V_\alpha\}$ and $\left\{W_\beta\right\}$ are distinct partitions of $p^{-1}(U)$. Fix an arbitrary α and let $e \in V_\alpha$. Then, there exists a unique β such that $e \in W_\beta$. We will prove that $V_\alpha = W_\beta$. Suppose on the contrary that $V_\alpha \neq W_\beta$. Then, $V_\alpha \setminus W_\beta \neq \emptyset$ and $V_\alpha \cap W_\beta \neq \emptyset$. So, $V_\alpha \cap W_\beta$ is open (intersection of two open sets is open), and

$$V_\alpha = (V_\alpha \cap W_\beta) \cup (V_\alpha \setminus W_\beta).$$

Let $x \in V_\alpha \setminus W_\beta$ (which is open), then there exists $\beta_x \neq \beta$ such that $x \in W_{\beta_x}$. Notice that

$$V_\alpha \setminus W_\beta = V_\alpha \cap \bigcup_{x \in V_\alpha \setminus W_\beta} W_{\beta_x}, \quad \text{which is open.}$$

Since U is connected and $p^{-1}(U)$ is homeomorphic to U through each slice, V_α must be connected as well, i.e. it cannot be decomposed into two disjoint open subsets, as would happen if $V_\alpha \setminus W_\beta$ were non-empty and open. This results in a contradiction. □

Lemma 1.12. Let $p : E \to B$ be a covering map, where B is connected. If

$$\left|p^{-1}(b_0)\right| = k \text{ for some } b_0 \in B, \quad \text{then} \quad \left|p^{-1}(b)\right| = k \text{ for all } b \in B.$$

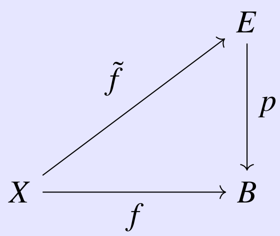
In such a case, we say that E is a k-fold covering of B.

Proof. By the local triviality property, the set of elements $b \in B$ such that $\left|p^{-1}(b)\right| = \left|p^{-1}(b_0)\right|$ is both open and closed. □

Proposition 1.9. Let $p : E \to B$ be a covering map. If B is Hausdorff, then E is also Hausdorff. The same applies if B is T_3, $T_{3\frac{1}{2}}$, or locally compact Hausdorff. In addition, if B is compact and $p^{-1}(b)$ is finite for all $b \in B$, then E is compact.

1.5. *Homotopy Lifting Property of Covering Maps*

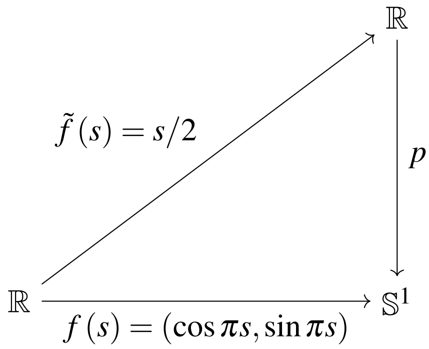
Definition 1.18. Let $p : E \to B$ be a B-space and $f : X \to B$ be a continuous map. A lifting of f to E over p is a continuous map $\widetilde{f} : X \to E$ such that $p \circ \widetilde{f} = f$.

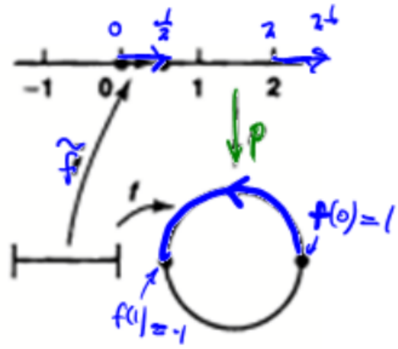


Example 1.21. Let $p : \mathbb{R} \to \mathbb{S}^1$ be a covering map. The path $f(s) = (\cos \pi s, \sin \pi s)$ lifts to the path $\widetilde{f}(s) = s/2$ beginning at 0 and ending at $1/2$. To verify this statement, consider

$$(p \circ \widetilde{f})(s) = p(s/2) \quad \text{so} \quad p(s) = f(2s) = (\cos 2\pi s, \sin 2\pi s)$$

which wraps around the unit circle anticlockwise once from $(1,0)$ to $(-1,0)$ as s increases from 0 to $1/2$.

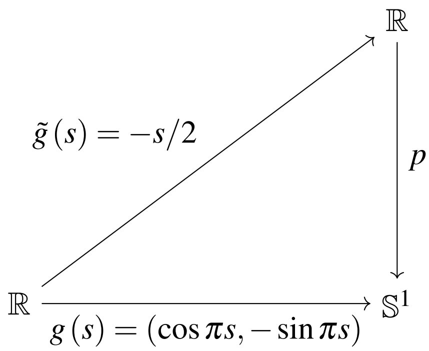


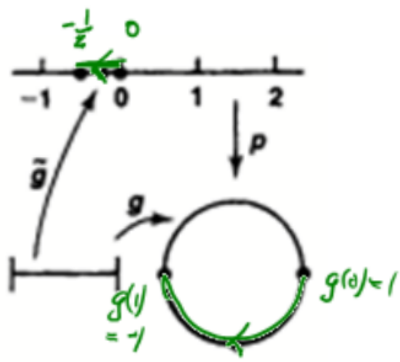


Example 1.22. Let $p : \mathbb{R} \to \mathbb{S}^1$ be a covering map. The path $f(s) = (\cos \pi s, -\sin \pi s)$ lifts to the path $\widetilde{g}(s) = -s/2$ beginning at 0 and ending at $-1/2$. Again, we shall verify this statement. Note that

$$(p \circ \widetilde{g})(s) = p(-s/2) \quad \text{so} \quad p(s) = f(-2s) = (\cos 2\pi s, \sin 2\pi s)$$

which wraps around the unit circle from $(1,0)$ to $(-1,0)$ clockwise as s decreases from 0 to $-1/2$.

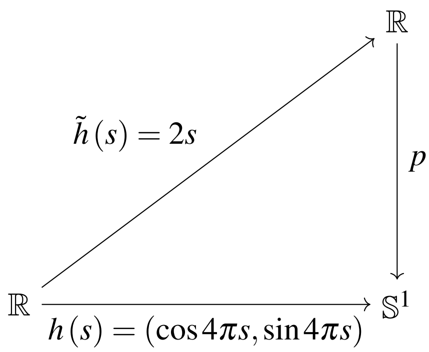


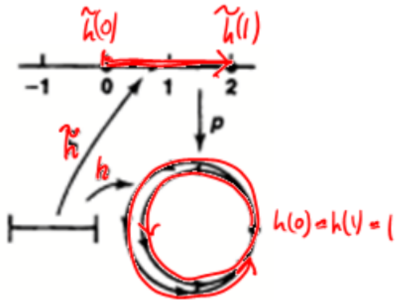


Example 1.23. Let $p : \mathbb{R} \to \mathbb{S}^1$ be a covering map. The path $h(s) = (\cos 4\pi s, \sin 4\pi s)$ lifts to the path $\widetilde{h}(s) = 2s$ beginning at 0 and ending at 2. Again, we shall verify this statement. Note that

$$\left(p \circ \widetilde{h}\right)(s) = p(2s) \quad \text{so} \quad p(s) = h(s/2) = (\cos 2\pi s, \sin 2\pi s)$$

which wraps around the unit circle from $(1,0)$ to itself in an anticlockwise manner as s increases from 0 to 2.





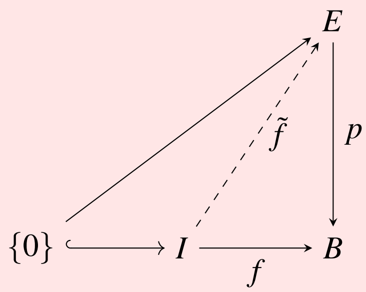
Lemma 1.13. Let $p : E \to B$ be a covering map and $f : X \to B$ be a continuous map. Suppose

- **(i)** there exists an open subset $U \subseteq B$ with $f(X) \subseteq U$ such that U is evenly covered by p;
- **(ii)** there exists a connected subset $X_0 \subseteq X$ such that there exists a lifting $\widetilde{f_0} : X_0 \to E$ of $f|_{X_0}$ to E

Then, there exists a lifting $\widetilde{f} : X \to E$ of f to E such that $\widetilde{f}|_{X_0} = \widetilde{f_0}$.

Theorem 1.8 (unique path lifting property)**.** Let $p : E \to B$ be a covering map and $f : I \to B$ be any path in B beginning at $b_0 = f(0)$. For any choice $e_0 \in p^{-1}(b_0)$ in the fiber over b_0, there exists

$$\text{a unique lifting } \widetilde{f} : I \to E \quad \text{to a path } E \text{ beginning at } e_0.$$



Theorem 1.9 (unique path-homotopy lifting property)**.** Let $p : E \to B$ be a covering map and $F : I \times I \to B$ be a continuous map such that $F(0,0) = b_0$. For any choice $e_0 \in p^{-1}(b_0)$ in the fiber over b_0, there exists

$$\text{a unique lifting } \widetilde{F} : I \times I \to E \quad \text{with } \widetilde{F}(0,0) = e_0.$$

Moreover, if F is a path homotopy, then $\widetilde{F}$ is a path homotopy.

Proposition 1.10. Let $p : E \to B$ be a covering map and f and g be two paths in B from b_0 to b_1. Let $e_0 \in p^{-1}(b_0)$ be a point in the fiber over b_0. Let $\widetilde{f}$ and $\widetilde{g}$ be their respective listings to paths in E beginning at e_0. If f and g are path homotopic in B, then $\widetilde{f}$ and $\widetilde{g}$ end at the same point $e_1 \in p^{-1}(b_1)$ and are path homotopic.

Lemma 1.14. Let $p : E \to B$ be a covering map. Let α and β be paths in B such that $\alpha(1) = \beta(0)$. Suppose $\widetilde{\alpha}$ and $\widetilde{\beta}$ are lifting of them such that $\widetilde{\alpha}(1) = \widetilde{\beta}(0)$. Then, $\widetilde{\alpha} * \widetilde{\beta}$ is a lifting of $\alpha * \beta$.

Definition 1.19 (the lifting correspondence)**.** Let $p : E \to B$ be a covering map. Let $b_0 \in B$ and $e \in p^{-1}(b_0)$. For any path-homotopy class α of paths in B beginning at b_0, there exists a unique path-homotopy class $\widetilde{\alpha}_{e_0}$ of paths in E beginning at e_0 lifting α.

Definition 1.20 (the monodromy action)**.** Let $p: E \to B$ be a covering map. Let $b_0 \in B$. For any $\alpha \in \pi_1(B, b_0)$, the lifting correspondence induced by α is a map from $p^{-1}(b_0)$ to itself, i.e.

$$- * \alpha : p^{-1}(b_0) \to p^{-1}(b_0), \quad e_0 \mapsto e_0 * \alpha$$

The monodromy action of $\pi_1(B, b_0)$ on $p^{-1}(b_0)$ is the map

$$* : p^{-1}(b_0) \times \pi_1(B, b_0) \to p^{-1}(b_0), \quad (e_0, \alpha) \mapsto e_0 * \alpha.$$

This is a well-defined right action of $\pi_1(B, b_0)$ on $p^{-1}(b_0)$.

Theorem 1.10. Let $p: E \to B$ be a covering map and $b_0 \in B$. Consider the monodromy action of $\pi_1(B, b_0)$ on $p^{-1}(b_0)$.

(i) If E is path-connected, the action is transitive;

(ii) If E is simply-connected, the action is simply transitive

Corollary 1.3 (Munkres p. 348 Question 8)**.** Let $p: E \to B$ be a covering map, where E is path-connected. If B is simply-connected, then p is a homeomorphism.

Proof. Since E is path-connected, then

$$\Phi : H \backslash \pi_1(B, b_0) \to p^{-1}(b_0) \quad \text{is bijective} \quad \text{for all } b_0 \in B.$$

Since B is simply-connected, then $\pi_1(B, b_0)$ is trivial, so $p^{-1}(b_0)$ contains exactly one element. Hence, p is a bijection. Lastly, as p is open, then p^{-1} is a continuous map, and it follows that p is a homeomorphism. □

1.6. *The Fundamental Group of the Circle*

Theorem 1.11 (fundamental group of $\mathbb{S}^1$)**.** The fundamental group of $\mathbb{S}^1$ is isomorphic to $\mathbb{Z}$. More precisely, consider the exponential covering map

$$p : \mathbb{R} \to \mathbb{S}^1 \quad \text{given by} \quad p(x) = (\cos 2\pi x, \sin 2\pi x).$$

Since $\mathbb{R}$ is simply-connected, the monodromy action of $\pi_1(\mathbb{S}^1, 1)$ on $\pi^{-1}(1) = \mathbb{Z}$ is simply transitive. Hence, we obtain a bijection

$$\phi : \pi_1(\mathbb{S}^1, 1) \to \mathbb{Z} \quad \text{where} \quad \alpha \mapsto 0 * \alpha.$$

Proof. We claim that ϕ is a group isomorphism, and hence an isomorphism of groups. Suppose $\alpha, \beta \in \pi_1(\mathbb{S}^1, 1)$. Then, we choose loops $f \in \alpha$ and $g \in \beta$ in $\mathbb{S}^1$ based at 1 representing α and β respectively. By the unique path lifting property, we obtain the liftings

$$\widetilde{f} \text{ of } f \text{ and } \widetilde{g} \text{ of } g \quad \text{to paths in } \mathbb{R} \text{ beginning at } 0.$$

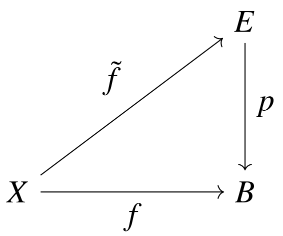
Let $n=\widetilde{f}(1)$ and $m=\widetilde{g}(1)$ in $p^{-1}(1)=\mathbb{Z}$. So, $0*\alpha=n$ and $0*\beta=m$ by definition.

Let $\widetilde{\widetilde{g}}$ be the path $n+\widetilde{g}(s)$ in $\mathbb{R}$ beginning at n. By the properties of the exponential map $p:\mathbb{R}\to\mathbb{S}^1$ (namely a group homomorphism with kernel $\mathbb{Z}$), it follows that $\widetilde{\widetilde{g}}$ is the unique lifting of g beginning at n.

Then, the product path $\widetilde{f}*\widetilde{\widetilde{g}}$ is defined, and it is the unique lifting of $f*g$ beginning at 0. The endpoint of this path is $\widetilde{\widetilde{g}}(1)=n+m$. Since $f*g$ represents $\alpha*\beta$, it follows that $0*(\alpha*\beta)=n+m$. $\square$

Example 1.24 (Munkres p. 347 Question 3)**.** Let $p:E\to B$ be a covering map. Suppose α and β are paths in B with $\alpha(1)=\beta(0)$; let $\widetilde{\alpha}$ and $\widetilde{\beta}$ be liftings of them such that $\widetilde{\alpha}(1)=\widetilde{\beta}(0)$. Prove that $\widetilde{\alpha}*\widetilde{\beta}$ is a lifting of $\alpha*\beta$.

Solution. Recall the following definition: if $p:E\to B$ is a covering map and $f:X\to B$ be a continuous map. A lifting of f to E over p is a continuous map $\widetilde{f}:X\to E$ such that $p\circ\widetilde{f}=f$.



As such, it suffices to prove that there exists a continuous map $\widetilde{\alpha}*\widetilde{\beta}:X\to E$ such that $p\circ\left(\widetilde{\alpha}*\widetilde{\beta}\right)=\alpha*\beta$. We are given that

$$p\circ\widetilde{\alpha}=\alpha \quad \text{and} \quad p\circ\widetilde{\beta}=\beta.$$

Since $\widetilde{\alpha}(1)=\widetilde{\beta}(0)$, we can concatenate the paths $\widetilde{\alpha}$ and $\widetilde{\beta}$. Since

$$(\alpha*\beta)(t)=\begin{cases}\alpha(2t) & \text{if } 0\le t\le 1/2;\\ \beta(2t-1) & \text{if } 1/2\le t\le 1\end{cases}, \quad \text{we can define} \quad \left(\widetilde{\alpha}*\widetilde{\beta}\right)=\begin{cases}\widetilde{\alpha}(2t) & \text{if } 0\le t\le 1/2;\\ \widetilde{\beta}(2t-1) & \text{if } 1/2\le t\le 1.\end{cases}$$

Hence,

$$\begin{aligned}p\circ\left(\widetilde{\alpha}*\widetilde{\beta}\right)&=p\circ\widetilde{\alpha}(2t)=(\alpha*\beta)(t) \quad \text{for } 0\le t\le 1/2 \quad \text{and}\\ p\circ\left(\widetilde{\alpha}*\widetilde{\beta}\right)&=p\circ\widetilde{\beta}(2t-1)=(\alpha*\beta)(t) \quad \text{for } 1/2\le t\le 1.\end{aligned}$$

Hence, $p\circ\left(\widetilde{\alpha}*\widetilde{\beta}\right)=\alpha*\beta$ for all $0\le t\le 1$. $\square$

Example 1.25 (Munkres p. 348 Question 6)**.** Consider the maps

$$g,h:\mathbb{S}^1\to\mathbb{S}^1 \quad \text{given by} \quad g(z)=z^n \text{ and } h(z)=1/z^n.$$

Here, $\mathbb{S}^1=\{z\in\mathbb{C}:|z|=1\}$. Compute the induced homomorphisms g_*,h_* of the infinite cyclic group $\pi_1(\mathbb{S}^1,b_0)$ into itself.

Solution. We have

$$g_*, h_* : \pi_1(\mathbb{S}^1, b_0) \to \pi_1(\mathbb{S}^1, b_0).$$

Recall that any loop around $\mathbb{S}^1$ can be represented by $\gamma(t) = e^{2\pi i t}$ for $0 \le t \le 1$. This corresponds to a generator of $\pi_1(\mathbb{S}^1, b_0)$, say $1 \in \mathbb{Z}$. So,

$$g(\gamma(t)) = e^{2n\pi i t},$$

for which the loop runs n times around the origin as t goes from 0 to 1. Under the induced homomorphism g_*, the generator 1 of $\pi_1(\mathbb{S}^1, b_0)$ is mapped to n, i.e. $g_*(1) = n$. Since $\pi_1(\mathbb{S}^1, b_0) \cong \mathbb{Z}$, then $g_* : \mathbb{Z} \to \mathbb{Z}$ is multiplication by n, i.e. for any $k \in \mathbb{Z}$, we have $g_*(k) = nk$.

In a similar fashion, $h_* : \mathbb{Z} \to \mathbb{Z}$ is multiplication by $-n$, i.e. for any $k \in \mathbb{Z}$, we have $h_*(k) = -nk$. □

1.7. *Retractions*

We now establish some classical results of Topology that follow from our knowledge of $\pi_1(\mathbb{S}^1, 1) \cong \mathbb{Z}$.

Definition 1.21 (retraction and retract)**.** Let X be a topological space and $A \subseteq X$ be a subspace. We have the canonical continuous inclusion map $j : A \to X$.

- A retraction of X onto A is a continuous map

$$r : X \to A \quad \text{such that} \quad r \circ j = \text{id}_A$$

- A is a retract of X if and only if there exists a retraction of X onto A

Example 1.26. A singleton $\{x_0\}$ is a retract of the unit disc B^2. In particular, we can consider the retraction $p(x_0) = x_0$ for all $x_0 \in B_2$.

Example 1.27. The unit circle $\mathbb{S}^1$ is a retract of the annulus. To see why, consider the following retraction: the inner ring of the annulus being *pushed outwards* such that it coincides with the outer ring. Alternatively, the outer ring can be *pushed inwards* such that it coincides with the inner ring. In either case, we would obtain $\mathbb{S}^1$.

Example 1.28. The unit circle $\mathbb{S}^1$ is a retract of the punctured disc.

Example 1.29. The unit circle $\mathbb{S}^1$ is a retract of the punctured plane $\mathbb{R}^2 \setminus \{\mathbf{0}\}$[†] via the retraction

$$r : \mathbb{R}^2 \setminus \{\mathbf{0}\} \to \mathbb{S}^1 \quad \text{via} \quad r(x) = x / \|x\|.$$

Lemma 1.15. If A is a retract of X, then $j_* : \pi_1(A, a) \to \pi_1(X, a)$ is injective.

[†] Here is an animation in Desmos

Proof. We have

$$\begin{aligned} r_* \circ j_* &= (r \circ j)_* && \text{by functoriality} \\ &= (\mathrm{id}_A)_* && \text{since there exists a retraction of } X \text{ onto } A \\ &= \mathrm{id}_{\pi_1(A,a)} && \text{by functoriality} \end{aligned}$$

This means that for any $[\alpha] \in \pi_1(A,a)$ (recall that $[\alpha]$ is a homotopy class of loops based at a in A), applying j_* to $[\alpha]$ first and then applying r_* returns $[\alpha]$. Also, note that $_* : X \to A$ is a retraction.

Hence, given any $[\alpha], [\beta] \in \pi_1(A,a)$, if $j_*([a]) = j_*([\beta])$, then applying r_* to both sides yields $[\alpha] = [\beta]$. □

Theorem 1.12 (no-retraction theorem). There is no retraction of B^2 onto S^1.

Proof. Suppose on the contrary that the unit circle $\mathbb{S}^1$ were a retract of the unit disc B^2. Then, there exists $j_* : \pi_1(\mathbb{S}^1, 1) \to \pi_1(B^2, 1)$ is injective. However,

$$\text{the fundamental group of } \mathbb{S}^1 \text{ is } \mathbb{Z} \quad \text{but} \quad \text{the fundamental group of } B^2 \text{ is trivial.}$$

By considering group isomorphism, we see that there is no injection j_* from $\mathbb{Z}$ to the trivial group, which is a contradiction. □

Corollary 1.4. The following hold:

$$\begin{aligned} \text{the identity map } i : \mathbb{S}^1 \to \mathbb{S}^1 &\quad \text{is not nulhomotopic in } \mathbb{S}^1 \quad \text{and} \\ \text{the inclusion map } j : \mathbb{S}^1 \to \mathbb{R}^2 \setminus \{\mathbf{0}\} &\quad \text{is not nulhomotopic in } \mathbb{R}^2 \setminus \{\mathbf{0}\} \end{aligned}$$

Lemma 1.16. Let $h : \mathbb{S}^1 \to X$ be a continuous map. Then, the following are equivalent:

(1) h is nulhomotopic in X

(2) h extends to a continuous map $k : B^2 \to X$, i.e. $k \circ j = h$ where $j : \mathbb{S}^1 \hookrightarrow B^2$

(3) h_* is the trivial homomorphism of fundamental groups, i.e. $\pi_1(\mathbb{S}^1, 1) \to \pi_1(X, h(1))$

Proof. We first prove **(1)** implies **(2)**. Let $H : \mathbb{S}^1 \times I \to X$ be a homotopy between h and a constant map. Let $\pi : \mathbb{S}^1 \times I \to B^2$ be the map $\pi(x,t) = (1-t)x$. Then, π is a quotient map as it collapses $\mathbb{S}^1 \times 1$ to the point $\mathbf{0}$ and is otherwise injective. Since H is constant on $\mathbb{S}^1 \times 1$, it induces a continuous map $k : B^2 \to X$ that is an extension of h via the quotient map π.

We then prove **(2)** implies **(3)**. Suppose $j : \mathbb{S}^1 \hookrightarrow B^2$ is the inclusion map. Then, $h = k \circ j$. However,

$$j_* : \pi_1(\mathbb{S}^1, b_0) \to \pi_1(B^2, b_0) = 1 \quad \text{is trivial.}$$

This is because the fundamental group of $\mathbb{S}^1$ is $\mathbb{Z}$ whereas the fundamental group of B^2 is trivial. Hence, $h_* = k_* \circ j_*$ is trivial by functoriality.

Lastly, we prove **(3)** implies **(1)**. We refer to the following figure:

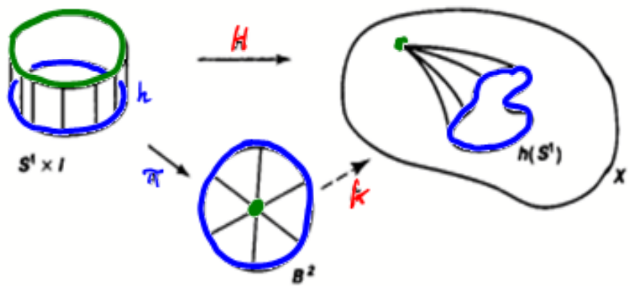


Figure 4: Visual proof that **(1)** implies **(2)**

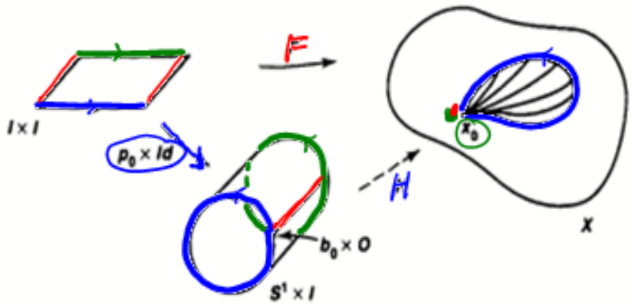


Figure 5: Visual proof that **(3)** implies **(1)**

Let $p_0 : I \to \mathbb{S}^1$ be a loop in $\mathbb{S}^1$ such that $[p_0]$ generates $\pi_1(\mathbb{S}^1, 1)$. Since h_* is trivial, the loop $f = h \circ p_0$ in X represents 1 in $\pi_1(X, x_0)$. Hence, there exists a path homotopy F in X between f and the constant path at x_0. The map

$$p_0 \times \text{id} : I \times I \to \mathbb{S}^1 \times I \quad \text{is a quotient map} \quad \text{for each } i \in I.$$

It maps $0 \times t$ and $1 \times t$ to $b_0 \times t$ but is otherwise injective. The path homotopy F maps $0 \times I$ and $1 \times I$ and $I \times 1$ to the point $x_0 \in X$, which induces a continuous map $H : \mathbb{S}^1 \times I \to X$ that is a homotopy between h and a constant map at x_0. □

Example 1.30 (Munkres p. 366 Question 6)**.** Show that a retract of a contractible space is contractible.

Solution. Intuitively, this is obvious. Let X be a contractible space and $f : X \times I \to X$ be the homotopy that continuously deforms X to a point, say $x_0 \in X$. Suppose $r : X \to Y$ is a retract of X, i.e. $r \circ i = \text{id}_Y$, where $i : Y \to X$ is the inclusion map. Hence, Y is a subspace of X and r restricts to the identity on Y. We wish to prove that Y is also contractible, so we shall construct a homotopy on Y using the given homotopy F on X.

Define the map $g : Y \times I \to Y$ by setting

$$g(y,t) = r(F(i(y),t)) \quad \text{for all } y \in Y \text{ and } t \in I.$$

This map is well-defined. We shall also check the boundary conditions. At $t = 0$, we see that $g(y,0) = r(F(i(y),0)) = r(i(y)) = y$ so $g(y,0) = y$ for all $y \in Y$. Also, at $t = 1$, we have $g(y,1) = r(F(i(y),1)) = r(x_0)$, where we recall that $x_0 \in X$ is the point that X contracts to. Hence,the image of $r(x_0)$ is some fixed point in Y, say $Y \ni y_0 = r(x_0)$.

Hence, g defines a homotopy between the identity map on Y and the constant map that sends every point in Y to y_0, showing that Y is contractible. □

Theorem 1.13 (Munkres p. 366 Question 7)**.** Let A be a subspace of X; let $j : A \to X$ be the inclusion map, and let $f : X \to A$ be a continuous map. Suppose there is a homotopy $H : X \times I \to X$ between the map $j \circ f$ and the identity map of X. Then, the following hold:

(a) If f is a retraction, then j_* is an isomorphism.
(b) If H maps $A \times I$ into A, then j_* is an isomorphism.

1.8. *Fixed Points*

Definition 1.22. A vector field on B^2 is a continuous map $v : B^2 \to \mathbb{R}^2$. v is said to be non-vanishing if and only if $v(x) \neq \mathbf{0}$ for all x. In such a case, v is a continuous map $v : B^2 \to \mathbb{R}^2 \setminus \{\mathbf{0}\}$.

Theorem 1.14. Given a non-vanishing vector field on B^2, i.e. $v : B^2 \to \mathbb{R}^2 \setminus \{\mathbf{0}\}$, there exists

$$\begin{aligned} &\text{a point of } \mathbb{S}^1 && \text{where the vector field points directly inward} \\ \text{and } &\text{a point of } \mathbb{S}^1 && \text{where the vector field points directly outward} \end{aligned}$$

That is to say,

$$\begin{aligned} &\text{there exists } p \in \mathbb{S}^1 && \text{such that } v(p) \in -\mathbb{R}_{>0} \cdot p \\ \text{and } &\text{there exists } q \in \mathbb{S}^1 && \text{such that } v(q) \in \mathbb{R}_{>0} \cdot q \end{aligned}$$

Proof. Suppose for any $p \in \mathbb{S}^1$, one has $v(p) \notin -\mathbb{R}_{>0} \cdot p$. Let $w : \mathbb{S}^1 \to \mathbb{R}^2 \setminus \{\mathbf{0}\}$ be the restriction of v to $\mathbb{S}^1$. Since w extends to a map of B^2 into $\mathbb{R}^2 \setminus \{\mathbf{0}\}$, it is nulhomotopic in $\mathbb{R}^2 \setminus \{\mathbf{0}\}$ (recall the difference between $\mathbb{S}^1$ and B^2 — the former denotes the unit circle, whereas the latter denotes the closed unit disc).

On the other hand,

$$F : \mathbb{S}^1 \times I \to \mathbb{R}^2 \quad \text{given by} \quad F(x,t) = tx + (1-t)w(x)$$

is

$$\text{a homotopy in } \mathbb{R}^2 \quad \text{from the inclusion map } j : \mathbb{S}^1 \to \mathbb{R}^2 \setminus \{\mathbf{0}\} \text{ to the map } w : \mathbb{S}^1 \to \mathbb{R}^2 \setminus \{\mathbf{0}\}.$$

We claim that for each $t \in I$ and $x \in \mathbb{S}^1$, we have $F(x,t) \neq \mathbf{0}$. This is clear for $t = 0$ and $t = 1$ by substituting these values into the formula for $F(x,t)$. We argue by contradiction that $F(x,t) \neq \mathbf{0}$ for all $t \in (0,1)$. Suppose on the contrary that $F(x,t) = \mathbf{0}$ for some $t \in (0,1)$. Then,

$$tx + (1-t)w(x) = 0 \quad \text{which implies} \quad v(x) = w(x) \in -\mathbb{R}_{>0} \cdot x$$

which is a contradiction to the hypothesis on v (not clear to me yet). Hence,

$$F \text{ is a homotopy in } \mathbb{R}^2 \setminus \{\mathbf{0}\} \quad \text{from } j : \mathbb{S}^1 \to \mathbb{R}^2 \setminus \{\mathbf{0}\} \text{ to } w : \mathbb{S}^1 \to \mathbb{R}^2 \setminus \{\mathbf{0}\}.$$

So, j is nulhomotopic in $\mathbb{R}^2 \setminus \{\mathbf{0}\}$, which is a contradiction (explain further). Continue the proof at home. □

Theorem 1.15 (Brouwer fixed-point theorem for the disc)**.** If $f : B^2 \to B^2$ is continuous,

$$\text{there exists a point } x \in B^2 \quad \text{such that } f(x) = x.$$

Proof. Suppose on the contrary that $f(x) \neq x$ for all $x \in B^2$. Then,

$$v : B^2 \to \mathbb{R}^2 \setminus \{\mathbf{0}\} \text{ given by} \quad v(x) = f(x) - x \quad \text{is a non-vanishing vector field on } B^2.$$

So, there exists $x \in \mathbb{S}^1$ such that $v(x) \in \mathbb{R}_{>0} \cdot x$ points directly outward, i.e. there exists $a \in \mathbb{R}_{>0}$ such that $f(x) - x = ax$, so $f(x) = (1+a)x$ would lie outside B^2 which is a contradiction. □

Remark 1.6. Here is a *funny remark* by Prof. Chin. If you place a map of Singapore on a table in our room such that the map is a scaled-down version of the area it represents, there exists a point on the map that corresponds exactly to the point in the room directly below it.

Note that $B^n \subseteq \mathbb{R}^n$.

Theorem 1.16 (Brouwer fixed-point theorem for B^n)**.** Let $f : B^n \to B^n$ be a continuous map. Then,

$$\text{there exists a point } x \in B^n \quad \text{such that } f(x) = x.$$

Corollary 1.5 (Perron-Frobenius theorem)**.** Let $\mathbf{A} \in \mathscr{M}_{3\times 3}(\mathbb{R}^+)$. Then, $\mathbf{A}$ has a positive real eigenvalue.

In fact, the Perron-Frobenius theorem holds for $n \times n$ matrices with positive real entries.

Proof. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation which represents multiplication by $\mathbf{A}$. Also, let

$$B = \mathbb{S}^2 \cap \{(x_1, x_2, x_3) : x_1, x_2, x_3 \geq 0\} \quad \text{which is homeomorphic to the ball } B^2.$$

If $\mathbf{x} = (x_1, x_2, x_3) \in B$, then

$$T(\mathbf{x}) \text{ is a vector} \quad \text{with all components being positive.}$$

This is because all entries of $\mathbf{A}$ are positive and at least one component of $\mathbf{x}$ is positive (since $\mathbf{x}$ lies on the sphere $\mathbb{S}^2$ so $\mathbf{x} \neq \mathbf{0}$). Hence, the map

$$\mathbf{x} \mapsto \frac{T(\mathbf{x})}{\|T(\mathbf{x})\|} \quad \text{is a continuous map of } B \text{ to itself.}$$

By the Brouwer fixed-point theorem, for $B \cong B^2$, B has a fixed point $\mathbf{x}_0$. Then, $T(\mathbf{x}_0) = \|T(\mathbf{x}_0)\| \, \mathbf{x}_0$, so T has the positive real eigenvalue $\|T(\mathbf{x}_0)\|$ with the corresponding eigenvector being $\mathbf{x}_0$. □

1.9. *Deformation Retracts and Homotopy Type*

Let X be a topological space and $A \subseteq X$ be a subspace. Then, we have the canonical continuous inclusion map $j : A \to X$.

Recall that a retraction of X onto A is a continuous map

$$r : X \to A \quad \text{such that} \quad r \circ j = \mathrm{id}_A \, .$$

In other words, $r|_A$ is the identity map of A. We say that A is a retract of X if and only if there exists a retraction of X onto A.

Definition 1.23 (deformation retraction)**.** A deformation retraction of X onto A is a continuous map $H : X \times I \to X$ such that

$$\text{for all } x \in X \quad \text{we have } H(x,0) = x \text{ and } H(x,1) \in A$$

and for all $t \in I$ and $a \in A$, one has $H(a,t) = a$, i.e. H fixes a throughout all time $t \in I$.

Equivalently, a deformation retraction is such that the map

$$\text{the map } r : X \to A \text{ defined by } r(x) = H(x,1) \quad \text{is a retraction of } X \text{ onto } A$$

and H is a homotopy between the identity map of X and the map $j \circ r$ such that each point of A remains fixed during the homotopy.

Hence,

$$A \text{ is a deformation retract of } X \quad \text{if and only if} \quad \text{there exists a deformation retraction of } X \text{ onto } A.$$

Example 1.31. A singleton is a deformation retract of $I = [0,1]$, the unit disc B^2 and $\mathbb{R}^2$.

Example 1.32. $\mathbb{S}^n$ is a deformation retract of $X = \mathbb{R}^{n+1} \setminus \{\mathbf{0}\}$. To see why, we have a continuous map

$$H : X \times I \to X \quad \text{such that} \quad H(x,t) = (1-t)x + tx/\|x\|$$

Example 1.33. A singleton is a retract, but not a deformation retract, of two distinct singletons. To see why, use connectedness.

Example 1.34. A singleton is a retract, but not a deformation retract, of a circle.

Example 1.35 (Munkres p. 366 Question 1)**.** Show that if A is a deformation retract of X and B is a deformation retract of A, then B is a deformation retract of X.

Solution. By definition, there exist continuous maps $G : X \times I \to X$ and $H : A \times I \to A$ such that

$$\begin{aligned} &\text{for all } x \in X \quad \text{we have } G(x,0) = x \text{ and } G(x,1) \in A \quad \text{and} \\ &\text{for all } a \in A \quad \text{we have } H(a,0) = a \text{ and } H(a,1) \in B \end{aligned}$$

Moreover, for all $t \in I$, $a \in A$, and $b \in B$, we have

$$G(a,t) = a \quad \text{and} \quad H(a,t) = b.$$

We need to construct a continuous map $F : X \times I \to X$ such that

$$\text{for all } x \in X \quad \text{we have } F(x,0) = x \text{ and } F(x,1) \in B.$$

Also, for all $t \in I$ and $b \in B$, we must have $F(b,t) = b$. The required continuous map is

$$F(x,t) = \begin{cases} G(x,2t) & \text{if } 0 \le t \le 1/2; \\ H(G(x,1),2t-1) & \text{if } 1/2 \le t \le 1. \end{cases}$$

We proceed with a verification — when $t = 1/2$, the first piece-wise component implies $F(x,1/2) = G(x,1) \in A$ whereas the second piece-wise component implies $F(x,1/2) = H(G(x,1),0)$. By the hypothesis, it was mentioned that $G(x,1) \in A$, so there exists $a_0 \in A$ such that $G(x,1) = a_0$. Hence, $H(G(x,1),0) = H(a_0,0) = a_0 \in A$, which corresponds with what we had earlier.

Moreover, when $t = 1$, we have $F(x,1) = H(G(x,1),1) = H(a_0,1) \in B$, which corresponds to the hypothesis as we require $F(x,1) \in B$. □

Lemma 1.17. Let $h,k : (X,x_0) \to (Y,y_0)$ be continuous maps. If h and k are homotopic and the image of the base point x_0 of X remains fixed at y_0 during the homotopy, then the homomorphism

$$h_*, k_* : \pi_1(X,x_0) \to \pi_1(Y,y_0) \quad \text{where} \quad [f] \mapsto [h \circ f], [k \circ f] \text{ are equal.}$$

Proof. By assumption, there is a homotopy $H : X \times I \to Y$ between h and k such that $H(x_0,t) = y_0$ for all t. If f is a loop in X based at x_0, the composite

$$I \times I \xrightarrow{f\times \mathrm{id}} X \times I \xrightarrow{H} Y \quad \text{is a path homotopy between } h\circ f \text{ and } k \circ f.$$

This is because H maps $x_0 \times I$ to y_0. □

Theorem 1.17. If A is a deformation retract of X and $x_0 \in A$, the inclusion map $j : (A,x_0) \to (X,x_0)$ induces

$$\text{an isomorphism} \quad j_* : \pi_1(A,x_0) \cong \pi_1(X,x_0) \text{ of fundamental groups.}$$

Corollary 1.6. The inclusion map

$$j : \mathbb{S}^n \to \mathbb{R}^{n+1} \setminus \{\mathbf{0}\} \quad \text{induces an isomorphism} \quad j_* : \pi_1(\mathbb{S}^1, b_0) \to \pi_1(\mathbb{R}^{n+1} \setminus \{\mathbf{0}\}, b_0)$$

of fundamental groups.

Example 1.36. The punctured xy-plane $(\mathbb{R}^2 \setminus \{\mathbf{0}\}) \times 0$ is a deformation retract of the space $\mathbb{R}^3 \setminus B$, where B is the z-axis in $\mathbb{R}^3$ via

$$H : (\mathbb{R}^3 \setminus B) \times I \to (\mathbb{R}^3 \setminus B), \quad \text{where} \quad H(x,y,z,t) = (x,y,(1-t)z).$$

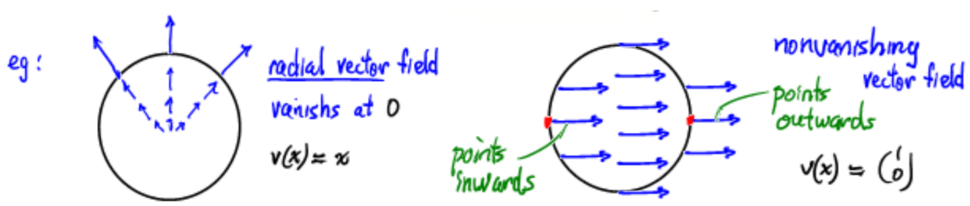


Figure 6: Vector field $\mathbf{v} : B^2 \to \mathbb{R}^2$

As shown in the figure above, we have $\pi_1(\mathbb{R}^2 \setminus \{\mathbf{0}\}, b_0) \cong \pi_1(\mathbb{R}^3 \setminus B, b_0)$. When $t = 0$, we have $H(x,y,z,0) = (x,y,z)$, which means H starts with the identity map on $\mathbb{R}^3 \setminus B$; when $t = 1$, we have $H(x,y,z,1) = (x,y,0)$, which collapses every point $(x,y,z) \in \mathbb{R}^3 \setminus B$ onto the plane $z = 0$.

Example 1.37 (figure-eight space)**.** The figure-eight space is a deformation retract of the doubly punctured plane $\mathbb{R}^2 \setminus \{p,q\}$ via a three-stage deformation.

Example 1.38 (theta space)**.** The theta space $\theta = \mathbb{S}^1 \cup (0 \times [-1,1])$ is a deformation retract of the doubly punctured plane $\mathbb{R}^2 \setminus \{p,q\}$. Intuitively, think of the theta space as the circle $\mathbb{S}^1$ with a vertical line through its centre.

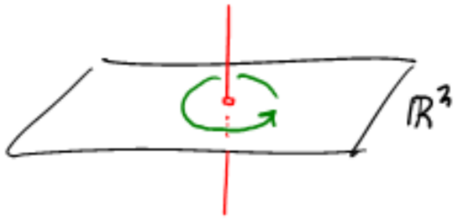


Figure 7: The punctured xy-plane is a deformation retract of $\mathbb{R}^3 \setminus B$

Remark 1.7. We infer that the figure-eight space and the theta space have isomorphic fundamental groups even though neither is a deformation retract of the other.

Definition 1.24 (homotopy inverse)**.** A homotopy inverse of a continuous map $f : X \to Y$ is

$$\text{a continuous map } g : Y \to X$$

such that

$$g \circ f : X \to X \text{ is homotopic to the identity map of } X \quad \text{and}$$
$$f \circ g : Y \to Y \text{ is homotopic to the identity map of } Y.$$

Then, f is a homotopy inverse of g.

Definition 1.25 (homotopy equivalence)**.** A homotopy equivalence from X to Y is a continuous map $f : X \to Y$ for which

$$\text{there eixsts a homotopy inverse } g : Y \to X.$$

X and Y are homotopy equivalent if and only if there exists a homotopy equivalence from X to Y.

Example 1.39. A homeomorphism is a homotopy equivalence.

Remark 1.8 (homotopy type)**.** Homotopy equivalence is an equivalence relation among topological spaces. Reflexivity and symmetry are obvious. To see why homotopy equivalence is transitive, suppose

$$f : X \to Y \text{ is a homotopy equivalence from } X \text{ to } Y \quad \text{and}$$
$$h : Y \to Z \text{ is a homotopy equivalence from } Y \text{ to } Z,$$

then $h \circ f : X \to Z$ is a homotopy equivalence from X to Z. An equivalence class for this equivalence relation is a homotopy type.

Example 1.40. Let X be a topological space and $A \subseteq X$ be a deformation retract of X. Then, the canonical inclusion $j : A \to X$ is a homotopy equivalence from A to X.

Example 1.41. The figure eight space and the theta space are deformation retracts of the doubly punctured plane $\mathbb{R}^2 \setminus \{p,q\}$ so they have the same homotopy type (or that they are homotopy equivalent spaces) even though neither is a deformation retract of the other.

Theorem 1.18. Let $f : X \to Y$ be a homotopy equivalence with $f(x_0) = y_0$. Then,

$$f_* : \pi_1(X,x_0) \to \pi_1(Y,y_0) \quad \text{is an isomorphism.}$$

In other words, $\pi_1(X,x_0)$ is a homotopy type invariant of (X,x_0).

Proof. Let $g : Y \to X$ be a homotopy inverse for f. Consider the maps and the induced homomorphism.

$$\begin{array}{ccc} (X,x_0) & \xrightarrow{\ f\ } & (Y,y_0) \\ {\scriptstyle g\circ f}\downarrow & \overset{g}{\swarrow} & \downarrow{\scriptstyle f\circ g} \\ (X,x_1) & \xrightarrow[\ f\ ]{} & (Y,y_1) \end{array} \qquad \begin{array}{ccc} \pi_1(X,x_0) & \xrightarrow{(f_{x_0})_*} & \pi_1(Y,y_0) \\ {\scriptstyle (g\circ f)_*}\downarrow & \overset{g_*}{\swarrow} & \downarrow{\scriptstyle (f\circ g)_*} \\ \pi_1(X,x_1) & \xrightarrow[(f_{x_1})_*]{} & \pi_1(Y,y_1) \end{array}$$

By the hypothesis,

$$g \circ f : (X,x_0) \to (X,x_1) \quad \text{is isomorphic to } \mathrm{id}_X : (X,x_0) \to (X,x_0).$$

So, there exists a path α in X such that

$$(g \circ f)_* = \widehat{\alpha} \circ (\mathrm{id}_X)_* = \widehat{\alpha}$$

so $g_* \circ (f_{x_0})_* = (g \circ f)_*$ is an isomorphism. Similarly, $(f_{x_1})_* \circ g_* = (f \circ g)_*$ is an isomorphism. Therefore, g_* is an isomorphism, and so is $(f_{x_0})_* = (g_*)^{-1} \circ \widehat{\alpha}$. □

Theorem 1.19 (Fuch's theorem). Two spaces X and Y have the same homotopy type if and only if there exists a single space Z such that

$$X \text{ and } Y \text{ are homeomorphic to deformation retracts of } Z.$$

Definition 1.26 (degree of continuous map). Let $h : \mathbb{S}^1 \to \mathbb{S}^1$ be a continuous map. We define the degree of h as follows: let $b_0 = (1,0) \in \mathbb{S}^1$. Choose a generator γ for the infinite cyclic group $\pi_1(\mathbb{S}^1, b_0)$. If x_0 is any point of $\mathbb{S}^1$, choose a path α in $\mathbb{S}^1$ from b_0 to x_0 and define $\gamma(x_0) = \widehat{\alpha}(\gamma)$. Then, $\gamma(x_0)$ generates $\pi_1(\mathbb{S}^1, x_0)$. The element $\gamma(x_0)$ is independent of the choice of the path α since the fundamental group of $\mathbb{S}^1$ is Abelian.

Choose $x_0 \in \mathbb{S}^1$ and let $h(x_0) = x_1$. Consider the homomorphism

$$h_* : \pi_1(\mathbb{S}^1, x_0) \to \pi_1(\mathbb{S}^1, x_0)$$

Since both groups are infinite cyclic, we have

$$h_*(\gamma(x_0)) = d\gamma(x_1) \quad \text{for some } d \in \mathbb{Z}.$$

We say that d is the degree of h and it is denoted by $\deg h$.

Note that the degree of h is independent of the choice of the generator γ; choosing the other generator would merely change the sign of both sides of the equation $h_*(\gamma(x_0)) = d\gamma(x_1)$.

Example 1.42 (Munkres p. 366 Question 9)**.** The following properties hold:

- **(a)** The degree d is independent of the choice of x_0.
- **(b)** If

$$h, k : \mathbb{S}^1 \to \mathbb{S}^1 \text{ are homotopic,} \quad \text{then} \quad \text{they have the same degree.}$$

- **(c)** We have $\deg(h \circ k) = (\deg h) \cdot (\deg k)$.
- **(d)** The degree of the constant map is 0, the degree of the identity map is 1, the degree of the reflection map $\rho(x_1, x_2) = (x_1, -x_2)$ is -1, and the degree of the n^{th} power map $h(z) = z^n$, where $z \in \mathbb{C}$, is n.
- **(e)** If $h, k : \mathbb{S}^1 \to \mathbb{S}^1$ have the same degree, they are homotopic.

1.10. *Fundamental Group of a Union (Seifert-van Kampen Theorem)*

Theorem 1.20 (special case of Seifert-van Kampen theorem)**.** Suppose $X = U \cup V$, where U and V are open and path-connected sets of X, with $U \cap V$ non-empty and path-connected. Suppose $x_0 \in U \cap V$, and let

$$i : U \to X \quad \text{and} \quad j : V \to X \quad \text{be inclusion mappings.}$$

Then, $\pi_1(X, x_0)$ is generated by the images of

$$i_* : \pi_1(U, x_0) \to \pi_1(X, x_0) \quad \text{and} \quad j_* : \pi_1(V, x_0) \to \pi_1(X, x_0).$$

That is, for any loop f in X based at x_0, there exist loops g_i in U or in V based at x_0 such that f is path homotopic in X to $(g_1 * (g_2 * (\ldots * g_n)))$.

Remark 1.9. The assumption that $U \cap V$ is path-connected in the previous theorem is an important assumption.

Proof. Let $f : I \to X$ be a loop in X based at x_0. Since $\{U, V\}$ is an open covering of X, apply the Lebesgue number lemma to choose finite subdivisions of I

$$0 = a_0 < a_1 < \ldots < a_n = 1 \quad \text{fine enough such that}$$

for each i, we have $f([a_{i-1},a_i])$ is contained in either U or V. If for any index i, we have $f(a_i) \notin U \cap V$, then $i \neq 0$ and $i \neq n$ since $f(a_0) = f(a_n) = x_0$ is in $U \cap V$, and both $f([a_{i-1},a_i])$ and $f([a_i,a_{i+1}])$ are contained in either U or V. Hence,

$$f(a_i) \in U \setminus V \text{ or } V \setminus U \quad \text{implies} \quad f([a_{i-1},a_{i+1}]) \text{ is contained in } U \text{ or } V.$$

For each index $1 \leq i \leq n$, define f_i to be the path in X given by the positive linear map of I onto $[a_{i-1},a_i]$ followed by f. We then choose a path α_i in $U \cap V$ from x_0 to $f(a_i)$ (this is possible since $U \cap V$ is path-connected) with α_0 and α_n being the constant path at x_0.

By setting $g_i = (\alpha_{i-1} * f_i) * \overline{\alpha_i}$. Then, g_i is a loop in U or in V based at x_0 and

$$[g_1] * [g_2] * \ldots [g_n] = [f_1] * [f_2] * \ldots [f_n] = f.$$

Remember to include the figure from CCW's notes/Munkres □

Corollary 1.7. Suppose $X = U \cup V$, where U and V are open sets of X with $U \cap V \neq \emptyset$ and path-connected. If U and V are simply-connected, then X is simply-connected.

Remark 1.10. Break up $\mathbb{S}^1$ into its upper half and lower half, so the union of these two halves yields $\mathbb{S}^1$. However, their intersection yields two singletons, so this is not connected. Hence, the previous corollary is not applicable.

Theorem 1.21. If $n \geq 2$, the n-sphere $\mathbb{S}^n$ is simply-connected.

Proof. Let

$$p = (0,\ldots,0,1) \text{ and } q = (0,\ldots,0,-1) \quad \text{denote the north pole and south pole of } \mathbb{S}^n \text{ respectively.}$$

Let $U = \mathbb{S}^n \setminus \{p\}$ and $V = \mathbb{S}^n \setminus \{q\}$ be punctured planes. Define

$$\text{the stereographic projection } f : \mathbb{S}^n \setminus \{p\} \to \mathbb{R}^n \quad \text{such that} \quad f(x) = f(x_1,\ldots,x_{n+1}) = \frac{1}{1-x_{n+1}}(x_1,\ldots,x_n).$$

f is a homomorphism with inverse given by

$$g : \mathbb{R}^n \to (\mathbb{S}^n \setminus \{p\}) \quad \text{where} \quad g(y) = g(y_1,\ldots,y_n) = (t(y)\cdot y_1,\ldots,t(y)\cdot y_n, 1-t(y)).$$

Here,

$$t(y) = \frac{2}{1+\|y\|^2}.$$

Thus, U and V are homeomorphic in $\mathbb{R}^n$ and simply-connected. Now, $U \cap V = \mathbb{S}^1 \setminus \{p,q\} \neq \emptyset$, homeomorphic to $\mathbb{R}^n \setminus \{0\}$ under stereographic projection, and hence path-connected for $n \geq 2$.

This implies every point of $\mathbb{R}^n \setminus \{0\}$ can be joined to a point of $\mathbb{S}^{n-1}$ via a straight-line path and $\mathbb{S}^{n-1}$ is path-connected for $n \geq 2$. □

1.11. *Fundamental Groups of Some Surfaces*

Theorem 1.22 (fundamental group of product). $\pi_1(X \times Y, x_0 \times y_0) \cong \pi_1(X, x_0) \times \pi_1(Y, y_0)$

To be precise, let

$$p : X \times Y \to X \quad \text{and} \quad q : X \times Y \to Y \quad \text{be projection mappings.}$$

Then, the homomorphisms

$$p_* : \pi_1(X \times Y, x_0 \times y_0) \to \pi_1(X, x_0) \quad \text{and} \quad q_* : \pi_1(X \times Y, x_0 \times y_0) \to \pi_1(Y, y_0)$$

induce the homomorphism

$$\Phi : \pi_1(X \times Y, x_0 \times y_0) \to \pi_1(X, x_0) \times \pi_1(Y, y_0),$$

where

$$[f] \mapsto \Phi([f]) = p_*([f]) \times q_*([f]) = [p \circ f] \times [q \circ f].$$

Then, Φ is an isomorphism.

Corollary 1.8 (fundamental group of torus). The fundamental group of the torus $\mathbb{T} = \mathbb{S}^1 \times \mathbb{S}^1$ is isomorphic to $\mathbb{Z} \times \mathbb{Z}$.

Example 1.43 (Munkres p. 375 Question 1). Compute the fundamental groups of the *solid torus* $\mathbb{S}^1 \times B^2$ and the product space $\mathbb{S}^1 \times \mathbb{S}^2$.

Solution. We first compute the fundamental group of the solid torus. Suppose $x_0 \in \mathbb{S}^1$ and $y_0 \in B^2$. Then,

$$\pi_1(\mathbb{S}^1 \times B^2, x_0 \times y_0) \cong \pi_1(\mathbb{S}^1, x_0) \times \pi_1(B^2, y_0) \cong \mathbb{Z} \times \{e\} = \mathbb{Z} \quad \text{where } \{e\} \text{ denotes the trivial group.}$$

We then compute the fundamental group of the product space $\mathbb{S}^1 \times \mathbb{S}^2$, which is isomorphic to $\mathbb{Z} \times \{e\} = \mathbb{Z}$. □

Corollary 1.9. $\pi_1\left(\mathbb{S}^1 \times \ldots \times \mathbb{S}^1, x_0\right) \cong \mathbb{Z}^{\oplus n}$; this is the fundamental group of the n-dimensional torus.

Example 1.44 (Munkres p. 366 Question 2). For each of the following spaces, the fundamental group is either trivial, infinite cyclic, or isomorphic to the fundamental group of the figure eight. Determine for each space which of the three alternatives holds.

(a) The *solid torus* $B^2 \times \mathbb{S}^1$.

(b) The torus $\mathbb{T}$ with a point removed

(c) The cylinder $\mathbb{S}^1 \times I$

(d) The infinite cylinder $\mathbb{S}^1 \times \mathbb{R}$

(e) $\mathbb{R}^3$ with non-negative x, y, and z axes deleted

The following subsets of $\mathbb{R}^2$:

(i) $\{x : \|x\| > 1\}$
(ii) $\{x : \|x\| \geq 1\}$
(iii) $\{x : \|x\| < 1\}$
(iv) $\mathbb{S}^1 \cup (\mathbb{R}_+ \times 0)$
(v) $\mathbb{S}^1 \cup (\mathbb{R}_+ \times \mathbb{R})$
(vi) $\mathbb{S}^1 \cup (\mathbb{R} \times 0)$
(vii) $\mathbb{R}^2 \setminus (\mathbb{R}_+ \times 0)$

Solution. We first discuss **(a)** to **(e)**.

(a) The fundamental group is $\mathbb{Z}$, which is infinite cyclic.

(b) The fundamental group of $\mathbb{T}$ with a point removed is isomorphic to that of the figure eight

(c) We can retract the cylinder onto one of its circular boundaries without altering its fundamental group, hence the fundamental group of the cylinder is isomorphic to that of the circle, which is $\mathbb{Z}$

(d) Same argument as **(d)**; the fundamental group is $\mathbb{Z}$

(e) The space retracts onto $\mathbb{S}^2 \setminus \{p,q,r\} \cong \mathbb{R}^2 \setminus \{s,t\}$

We then discuss **(i)** to **(vii)**.

(i) The fundamental group is isomorphic to $\mathbb{Z}$

(ii) Same explanation as **(ii)**

(iii) Trivial group

(iv) The fundamental group is isomorphic to $\mathbb{Z}$ since $\mathbb{R}^+ \times 0$ can be retracted to $(1,0)$

(v) The fundamental group is isomorphic to $\mathbb{Z}$

(vi) The fundamental group is isomorphic to the figure eight as $\mathbb{R} \times 0$ can be retracted to the line segment from $(-1,0)$ to $(1,0)$ which is homotopic to the figure eight

(vii) Trivial group □

Definition 1.27 (real projective space)**.** The real projective n-space $\mathbb{RP}^n$ is the quotient space obtained from $\mathbb{S}^n$ by identifying each point $x \in \mathbb{S}^n$ with its antipode $-x \in \mathbb{S}^n$. That is,

$$\mathbb{RP}^n = \mathbb{S}^n / \{\pm 1\}$$

Definition 1.28 (topological n-manifold)**.** Let X be a topological space. We say that X is an n-manifold if

(i) X is locally homeomorphic to $\mathbb{R}^n$, i.e. for any $x \in X$, there exists an open neighbourhood $U \subseteq X$ of x such that U is homeomorphic to $\mathbb{R}^n$

(ii) X is Hausdorff

(iii) X is second-countable (countable basis for its topology)

Theorem 1.23 (covering map from sphere to real projective space)**.** $\mathbb{RP}^n$ is a compact topological n-manifold. The quotient map $p : \mathbb{S}^n \to \mathbb{RP}^n$ is a covering map.

Example 1.45. A surface is a connected topological 2-manifold.

Example 1.46 (fundamental group of projective space)**.** $\pi_1(\mathbb{RP}^2, y)$ is a group of order 2.

To see why, consider the projection $p : \mathbb{S}^2 \to \mathbb{RP}^2$ which is a covering map. Since $\mathbb{S}^2$ is simply-connected, then there exists a bijective correspondence between $\pi_1(\mathbb{RP}^2, y)$ and $p^{-1}(y)$. Since this is a two-element set, then $\pi_1(\mathbb{RP}^2, y)$ is a group of order 2. Any group of integers is isomorphic to $\mathbb{Z}/2$, i.e. the integers modulo 2.

Example 1.47 (Munkres p. 375 Question 2)**.** Let X be the quotient space obtained from B^2 by identifying each point $x \in \mathbb{S}^1$ with its antipode $-x$. Show that X is homeomorphic to the projective plane $\mathbb{RP}^2$.

Solution. Consider X as constructed from B^2 identified with the closed upper hemisphere of $\mathbb{S}^2$. Let

$$p : \mathbb{S}^2 \to \mathbb{RP}^2 \quad \text{and} \quad q : B^2 \to X \quad \text{be quotient maps}$$

and $\pi : \mathbb{S}^2 \to B^2$ be such that

$$\pi(x) = \begin{cases} x & \text{if } x \in B^2; \\ -x & \text{if } x \notin B^2. \end{cases}$$

This is a quotient map as $U \subseteq \mathbb{S}^2$ implies $\pi(U)$ is open and $V \subseteq B^2$ is open implies $\pi^{-1}(V) = V \cup (-V)$ is open. We construct the following commutative diagram:

$$\begin{array}{ccc} \mathbb{S}^2 & \xrightarrow{p} & \mathbb{RP}^2 \\ \pi\downarrow & & \downarrow r \\ B^2 & \xrightarrow{q} & X \end{array}$$

As $q \circ \pi$ is a quotient map, then $r : \mathbb{RP}^2 \to X$ is also a quotient map. Since for every $x \in X$, we have $(q \circ \pi)^{-1}(x) = \{x, -x\} \in \mathbb{RP}^2$ and every equivalence class $\{x, -x\}$ can be regarded as the inverse image of this type, then r is a bijection. This implies $\mathbb{RP}^2 = \left\{(q \circ \pi)^{-1}(x) : x \in X\right\}$. We conclude that r is a homeomorphism. □

Example 1.48 (Munkres p. 375 Question 4)**.** The space $\mathbb{RP}^1$ and the covering map $p : \mathbb{S}^1 \to \mathbb{RP}^1$ are familiar ones. What are they?

Solution. Think of the space $\mathbb{RP}^1$ as the set of lines through the origin in $\mathbb{R}^2$. In other words, this is the space of equivalence classes of points on the circle $\mathbb{S}^1$, where two points on $\mathbb{S}^1$ are *equivalent* if and only if they are antipodal (will formally define this when discussing the Borsuk-Ulam theorem). As such, the covering map p gives a natural way to relate points on the circle $\mathbb{S}^1$ to lines through the origin in $\mathbb{R}^2$. □

Lemma 1.18 (fundamental group of figure eight)**.** The fundamental group of the figure eight space is not abelian.

Proof. Let X denote the union of two circles A and B in $\mathbb{R}^2$ whose intersection is the single point x_0. We shall describe a certain covering space E of X.

The space E is the subspace of the plane consisting of the x-axis and the y-axis, along with tiny circles tangent to these axes,

$$\text{one circle tangent to the } x\text{-axis} \quad \text{at each non-zero integer point}$$

and

$$\text{one circle tangent to the } y\text{-axis} \quad \text{at each non-zero integer point}$$

As shown in Figure 9, the projection map $p : E \to X$ wraps the x-axis around the circle A and wraps the y-axis around the other circle B; in each case, the integer points are mapped by p onto the base point x_0. Each circle tangent to an integer point on the x-axis is mapped homeomorphcially by p onto B while each circle tangent to an integer point on the y-axis is mapped homeomorphically onto A; in each case, the point of tangency is mapped onto the point x_0. Note that each p is a covering map.

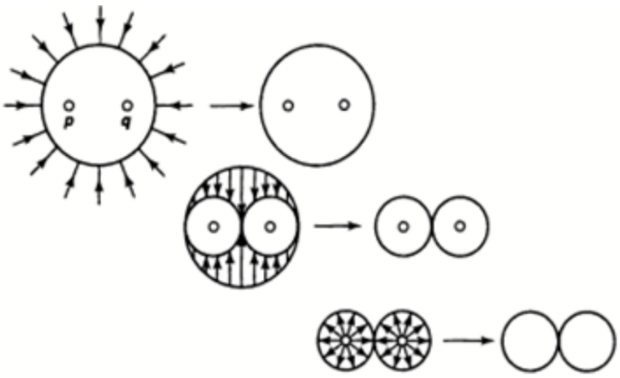


Figure 8: The figure eight space is a deformation retract of the doubly punctured plane

To express the above in terms of equations, we can do so as follows: let

$$\widetilde{f} : I \to E \text{ be the path } \widetilde{f}(s) = s \times 0 \quad \text{going along the } x\text{-axis from } (0,0) \text{ to } 1 \times 0.$$

Similarly, let

$$\widetilde{g} : I \to E \text{ be the path } \widetilde{g}(s) = 0 \times s \quad \text{going along the } y\text{-axis from } (0,0) \text{ to } 0 \times 1.$$

Let $f = p \circ \widetilde{f}$ and $g = p \circ \widetilde{g}$, then f and g are loops in the figure eight based at x_0 going around the circles A and B respectively. It suffices to prove that

$$f * g \text{ and } g * f \quad \text{are not path homotopic.}$$

To see why, we shall lift each of these to a path in E beginning at the origin. The path $f * g$ lifts to a path that goes along the x-axis from the origin to 1×0 and then goes once around the circle tangent to the x-axis at 1×0. On the other hand, the path $g * f$ lifts to a path in E that goes along the y-axis from the origin to 0×1, and then goes once around the circle tangent to the y-axis at 0×1. Since the lifted paths do not end at the same point, then $f * g$ and $g * f$ are not path homotopic. □

Example 1.49 (fundamental group of figure eight). We shall compute the fundamental group of the figure eight space X, which can be regarded as the wedge sum of two circles $S^1 \wedge S^1$. Let $U, V = S^1$ which intersect at a single point x_0. So, $X = U \cup V = S^1 \cup S^1$ and $U \cap V = \{x_0\}$.

Since it is a well-known fact that the fundamental group of a circle based at x_0 is the integers $\mathbb{Z}$, then $\pi_1(U, x_0) = \pi_1(V, x_0) \cong \mathbb{Z}$. Also, the fundamental group of a singleton is the trivial group $\{e\}$. So, $\pi_1(U \cap V, x_0) = \{e\}$. By the Seifert-Van Kampen theorem, we have the isomorphism

$$\pi_1(X, x_0) \cong \pi_1(U, x_0) * \pi_1(V, x_0)/N.$$

Recall that N is the least normal subgroup of the free product that contains all elements represented by words of the form $(i_1(g))^{-1} i_2(g)$, where $g \in \pi_1(U \cap V, x_0)$. Since $\pi(U \cap V, x_0) = \{e\}$, then any normal subgroup of this fundamental group is also trivial, for which it follows that $\pi_1(X, x_0) \cong \mathbb{Z} * \mathbb{Z} \cong \langle a, b \rangle = F_2$. Here,

$$\langle a, b \rangle \quad \text{denotes the free group generated by } a \text{ and } b, \quad \text{or equivalently,} \quad \text{the free group on two letters} = F_2.$$

So, the fundamental group of the figure eight space X is F_2.

Example 1.50 (Munkres p. 375 Question 3). Let X be the union of two circles A and B in $\mathbb{R}^2$ (figure eight space). Define $p : E \to X$ to be the map which wraps the x-axis around circle A and wraps the y-axis around the other circle B. Let E' be the subspace of E that is the union of the x-axis and the y-axis. Show that $p|_{E'}$ is not a covering map.

Solution. Consider the base point x_0, which is the centre of the figure eight space. A neighbourhood $x_0 \in U$ contains the union V of open intervals in A and B whose intersection is exactly $\{x_0\}$.

So, $(p|_{E'})^{-1}(V)$ is equal to the union of open intervals around integers on the x-axis and the union of open intervals around integers on the y-axis. However, none of these intervals is homeomorphic to V since removing an integer in an interval gives two connected components, while removing its image in V gives four connected components. It follows that $p|_{E'}$ is not a covering map. □

Definition 1.29 (double torus). The double torus $\mathbb{T}\#\mathbb{T}$ is the surface obtained by taking two copies of the torus $\mathbb{T}$, deleting a small open disc from each of them, and pasting the remaining pieces together along their edges.

Theorem 1.24. The fundamental group of the double torus $\mathbb{T}\#\mathbb{T}$ is non-abelian.

Proof. The figure eight is a retract of $\mathbb{T}\#\mathbb{T}$. □

1.12. *The Borsuk-Ulam Theorem*

Definition 1.30 (antipode-preserving map). A map $h : \mathbb{S}^n \to \mathbb{S}^m$ between spheres is antipode-preserving if and only if

$$h(-x) = -h(x) \quad \text{for all } x \in \mathbb{S}^n.$$

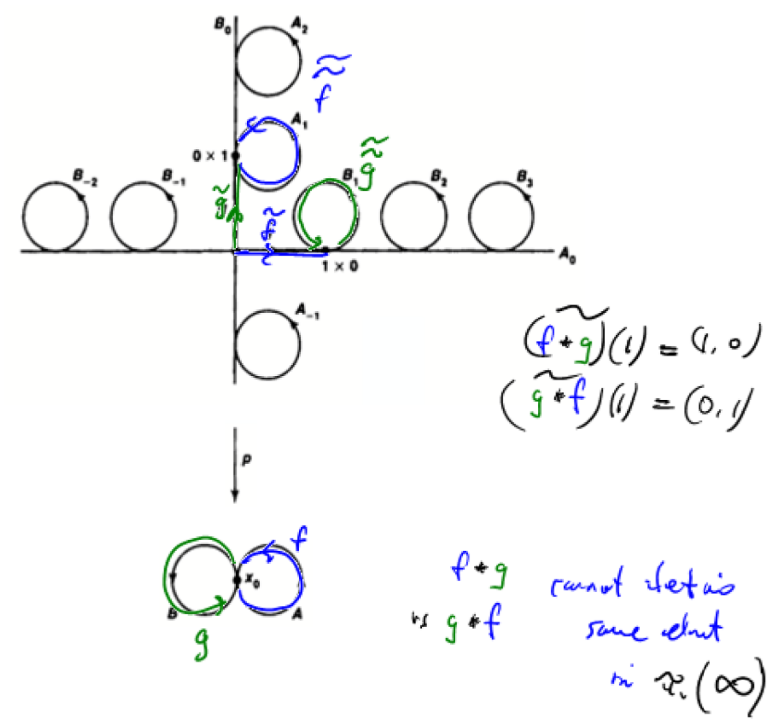


Figure 9: Fundamental group of figure eight

Theorem 1.25. For any $n \in \mathbb{Z}^+$, if $h : \mathbb{S}^n \to \mathbb{S}^n$ (i.e. map from n-sphere to itself) is continuous and antipode-preserving, then h is of odd degree and hence not nulhomotopic.

Theorem 1.26. If $\mathbb{S}^1 \to \mathbb{S}^1$ is continuous and antipode-preserving, then $h_* : \pi_1(\mathbb{S}^1, 1) \to \pi(\mathbb{S}^1, 1)$ is non-trivial, and hence h is not nulhomotopic.

Proof. We shall regard $S^1 \leq \mathbb{C}^\times$. First, assume that $h(1) = 1$. Let $\rho : \mathbb{S}^1 \to \mathbb{S}^1$ be any rotation of $\mathbb{S}^1$ sending $h(1) \in \mathbb{S}^1$ to $1 \in \mathbb{S}^1$. Note that ρ is antipode-preserving, hence $\rho \circ h$ is also antipode-preserving and $\rho \circ h$ fixes $1 \in \mathbb{S}^1$.

Also, if H is a homotopy between h and a constant map, then $\rho \circ H$ is a homotopy between $\rho \circ h$ and a constant map. Hence, $\rho \circ H$ is not nulhomotopic, which implies h is not nulhomotopic.

Let $q : \mathbb{S}^1 \to \mathbb{S}^1$ be the *squaring map* defined by $q(z) = z^2$. Note the following properties:

- q is a 2-fold covering map since every non-zero complex number has two distinct square roots
- q is a quotient map with $q^{-1}(z) = \{\pm z\}$, i.e. q identifies $\mathbb{S}^1 / \{\pm 1\} = \mathbb{RP}^1$ with $\mathbb{S}^1$
- Since h is continuous and antipode-preserving, it induces a continuous map $k : \mathbb{S}^1 \to \mathbb{S}^1$ such that $k \circ q = q \circ h$, i.e. the following diagram commutes:

$$\begin{array}{ccc} \mathbb{S}^1 & \xrightarrow{h} & \mathbb{S}^1 \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle q} \\ \mathbb{S}^1 & \xrightarrow{k} & \mathbb{S}^1 \end{array}$$

Next, we prove that

$$k_* : \pi_1(\mathbb{S}^1, 1) \to \pi_1(\mathbb{S}^1, 1) \quad \text{is non-trivial.}$$

If $\widetilde{f}$ is a path in $\mathbb{S}^1$ from 1 to -1, then $f = q \circ \widetilde{f}$ is a loop in $\mathbb{S}^1$ based at 1. Also, $h \circ \widetilde{f}$ is also a path in $\mathbb{S}^1$ from 1 to -1 as $h(1) = 1$ and h is antipode-preserving. Also, $[f]$ is a non-trivial element of $\pi_1(\mathbb{S}^1, 1)$ as $\widetilde{f}$ is a lifting of f over the covering map q that begins at 1 but does not end at 1.

Hence,

$$k_*[f] = [k \circ (q \circ \widetilde{f})] = [q \circ (h \circ \widetilde{f})] \quad \text{is a non-trivial element of } \pi_1(\mathbb{S}^1, 1)$$

because $h \circ \widetilde{f}$ is a lifting of $q \circ (h \circ \widetilde{f})$ over the covering map q that begins at 1 but does not end at 1.

We then claim that

$$h_* : \pi_1(\mathbb{S}^1, 1) \to \pi_1(\mathbb{S}^1, 1) \quad \text{is non-trivial.}$$

This follows from the fact that the following square commutes (due to functoriality):

$$\begin{array}{ccc} \pi_1(\mathbb{S}^1, 1) & \xrightarrow{h_*} & \pi_1(\mathbb{S}^1, 1) \\ {\scriptstyle q_*}\downarrow & & \downarrow{\scriptstyle q_*} \\ \pi_1(\mathbb{S}^1, 1) = \mathbb{Z} & \xrightarrow{k_*} & \pi_1(\mathbb{S}^1, 1) = \mathbb{Z} \end{array}$$

k_* is injective because it is a non-trivial homomorphism from $\mathbb{Z}$ to $\mathbb{Z}$. Also, q_* is injective as q is a covering map. The result follows. □

Theorem 1.27. For any positive integers m and n such that $m > n$, there does not exist a continuous and antipode-preserving map

$$g : \mathbb{S}^m \to \mathbb{S}^n \quad \text{from the } m\text{-sphere to the } n\text{-sphere.}$$

Corollary 1.10. There does not exist a continuous and antipode-preserving map

$$g : \mathbb{S}^2 \to \mathbb{S}^1 \quad \text{from the 2-sphere to the 1-sphere.}$$

Proof. Suppose on the contrary that $g : \mathbb{S}^2 \to \mathbb{S}^1$ is a continuous and antipode-preserving map. Let $j : \mathbb{S}^1 \to \mathbb{S}^2$ be the inclusion map of the equator, which is also continuous and antipode-preserving.

Then, $h = g \circ j$ is a continuous and antipode-preserving map $\mathbb{S}^1 \to \mathbb{S}^1$. So, h is not nulhomotopic. Define E to be the upper hemisphere of $\mathbb{S}^2$. Then, $g|_E$ is a continuous extension of h to $E \cong B^2$, so h is nulhomotopic, which is a contradiction. As such, such a g cannot exist. □

Theorem 1.28 (Borsuk-Ulam theorem). For any $n \in \mathbb{Z}_{\geq 0}$ and any continuous map $f : \mathbb{S}^{n+1} \to \mathbb{R}^{n+1}$, there exists $x \in \mathbb{S}^n$ such that $f(x) = f(-x)$ in $\mathbb{R}^{n+1}$.

Figure 10: Fundamental group of double torus

Corollary 1.11. Any continuous map $f: \mathbb{S}^{n+1} \to \mathbb{R}^{n+1}$ is not injective.

Theorem 1.29 (Borsuk-Ulam theorem for $\mathbb{S}^2$)**.** For any continuous map $f: \mathbb{S}^2 \to \mathbb{R}^2$, there exists $x \in \mathbb{S}^2$ such that $f(x) = f(-x)$ in $\mathbb{R}^2$.

Proof. Suppose on the contrary that $f: \mathbb{S}^2 \to \mathbb{R}^2$ is a continuous map such that $f(x) = f(-x)$ for all $x \in \mathbb{S}^2$. Then, the trick is to define $\mathbb{S}^2 \to \mathbb{S}^1$ given by

$$g(x) = \frac{f(x) - f(-x)}{\|f(x) - f(-x)\|}$$

is a continuous and antipode-preserving map, which contradicts an earlier theorem (or rather, corollary) that there does not exist a continuous and antipode-preserving map from the 2-sphere to the 1-sphere. □

Remark 1.11. Quote by CCW: *I mean it's kind of cute — this is the nature of topological results* when relating the Borsuk-Ulam theorem to the following theorem by Lusternik and Schnirelmann.

Theorem 1.30 (Lusternik-Schnirelmann theorem)**.** For $n \geq 1$, suppose $\mathbb{S}^n$ is covered by $n+1$ many closed subsets. Then, at least one of them contains a pair of antipodal points.

Proof. Suppose $A_0, A_1, \ldots, A_n \subseteq \mathbb{S}^n$ are $n+1$ closed subsets and

$$\bigcup_{i=0}^{n} A_i = \mathbb{S}^n.$$

Consider the map $f: \mathbb{S}^n \to \mathbb{R}^n$ given by

$$f(x) = (d(x, A_1), \ldots, d(x, A_n)) \quad \text{for all } x \in \mathbb{S}^n.$$

Here,

$$d(x, A_i) = \inf\{d(x,y) \in \mathbb{R}_{\geq 0} : y \in A_i\}.$$

Note that f is a continuous real-valued function of $x \in \mathbb{S}^n$ and

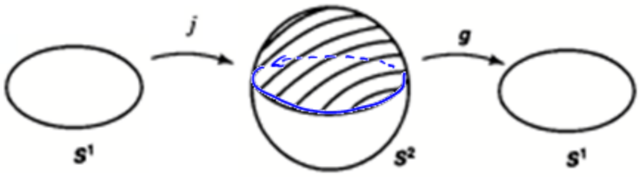
$$d(x, A_i) = 0 \quad \text{if and only if } x \in A_i \quad \text{since } A_i \text{ is closed.}$$

By the Borsuk-Ulam theorem, there exists a pair of antipodal points $x, -x \in \mathbb{S}^n$ such that $f(x) = f(-x)$ in $\mathbb{R}^n$, i.e. for all $1 \leq i \leq n$, we have $d(x, A_i) = d(-x, A_i)$.

If there exists $1 \leq i \leq n$ such that the above holds, then we have $x, -x \in A_i$ and we are done. Suppose otherwise, then for all $1 \leq i \leq n$, we have $x \notin A_i$ and $-x \notin A_i$. So, $0 = x + (-x) \in A_0$ and the result follows as well. □

Theorem 1.31 (bisection theorem in $\mathbb{R}^n$). For any non-negative integer n and n many bounded Jordan-measurable sets (boundary has Lebesgue measure 0 such as polyhedral regions) in $\mathbb{R}^n$, there exists a hyperplane of dimension $n-1$ in $\mathbb{R}^n$ that bisects them all.

Theorem 1.32 (bisection theorem in $\mathbb{R}^2$). For any two bounded polygonal regions in $\mathbb{R}^2$, there exists a line (a special hyperplane) in $\mathbb{R}^2$ that bisects each of them with respect to 2-dimensional area.



Proof. We take two bounded polygonal regions A_1 and A_2 in the plane $\mathbb{R}^1 \times 1$ in $\mathbb{R}^3$ and show that there exists a line L in this plane that bisects each of them.

Given a point $u \in \mathbb{S}^2$, we shall consider the plane P in $\mathbb{R}^3$ passing through the origin that has u has its unit normal vector. This plane divides $\mathbb{R}^3$ into two half-spaces;

$$\text{let } f_i(u) \quad \text{be the area of the portion of } A_i$$

that lies on the same side of P as does the vector u.

If u is the unit vector $\mathbf{k}$, then $f_i(u)$ is the area of A_i; if $u = -\mathbf{k}$, then $f_i(u) = 0$. Otherwise, the plane P intersects the plane $\mathbb{R}^2 \times 1$ in a line L that splits $\mathbb{R}^2 \times 1$ into two half-planes, and $f_i(u)$ is the area of that part of A_i that lies on one side of this line.

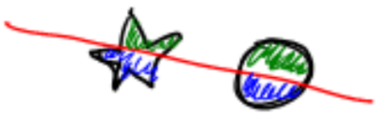


Figure 11: Bisection theorem in $\mathbb{R}^2$

Replacing u by $-u$ gives us the same plane, but the other half-space, so that $f_i(-u)$ is the area of that part of A_i that lies on the other side of P from u. It follows that

$$f_i(u) + f_i(-u) = \text{area of } A_i.$$

Now, consider the map

$$F : \mathbb{S}^2 \to \mathbb{R}^2 \quad \text{defined by} \quad F(u) = (f_1(u), f_2(u)).$$

The Borsak-Ulam theorem gives us a point $u \in \mathbb{S}^2$ for which $F(u) = F(-u)$. Then, $f_i(u) = f_i(-u)$ for $i = 1,2$ so that $f_i(u)$ is $\frac{1}{2} \cdot$ area A_i as desired. □

2. The Cauchy Integral Formula

2.1. *The Winding Number of a Simple Closed Curve*

Recall the definition of an induced homomorphism on fundamental groups, i.e.

$$h_* : \pi_1(X, x_0) \to \pi_1(Y, y_0).$$

This is defined to be the map that takes a loop in X based at x_0 and applies h to produce a loop in Y based at y_0.

Definition 2.1 (winding number)**.** Let $h : \mathbb{S}^1 \to \mathbb{R}^2 \setminus \{(0,0)\}$ be a continuous map. Suppose the induced homomorphism h_* carries a generator of the fundamental group of $\mathbb{S}^1$ to some integral power of a generator of the fundamental group of $\mathbb{R}^2 \setminus \{(0,0)\}$. Then, the integral power n is called

the winding number of h with respect to the origin $(0,0)$.

Geometrically, one can think of the winding number of a continuous map h as the number of times h wraps the circle $\mathbb{S}^1$ around the origin $(0,0)$; its sign depends on the choice of generators.

Example 2.1 (winding numbers of some closed curves)**.** Consider the following curves in $\mathbb{R}^2$ as shown in Figure 13. Recall some basic topology from MA2104 and/or MA3209 that these curves are simple as they loop back onto themselves, forming continuous shapes with no open ends. However, they are not simple because they intersect themselves at multiple points.

One can easily check that the winding numbers of these closed curves are ± 2 and 0 respectively.

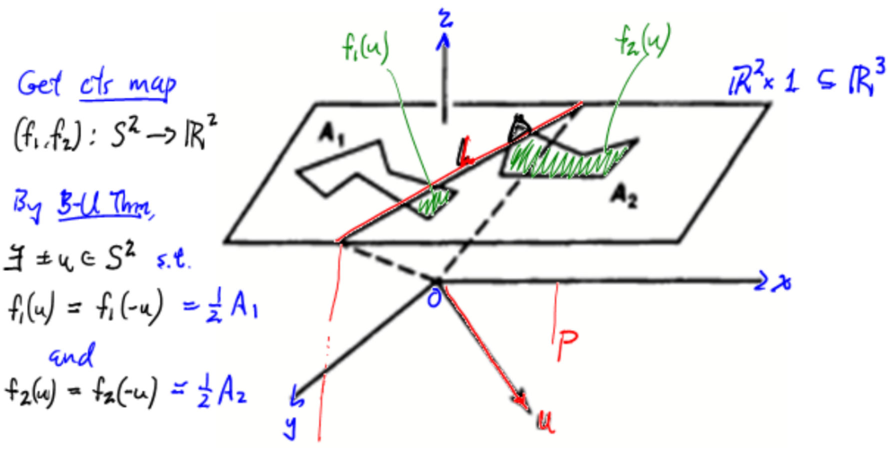


Figure 12: Bisection theorem in $\mathbb{R}^2$

Now, we ask the question: what can one say about the winding number of h if h is injective, i.e.

$$h \text{ is a homeomorphism of } \mathbb{S}^1 \quad \text{with a simple closed curve } C \text{ in } \mathbb{R}^2 \setminus \{(0,0)\}?$$

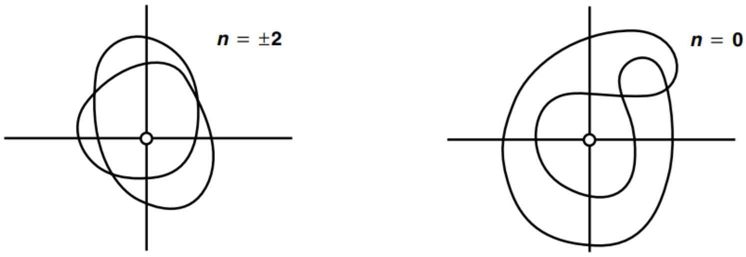


Figure 13: Winding numbers of some closed curves in $\mathbb{R}^2$

Figure 14 suggests that if $(0,0)$ belongs to the unbounded component of $\mathbb{R}^2 \setminus C$, then $n = 0$; if $(0,0)$ belongs to the bounded component of $\mathbb{R}^2 \setminus C$, then $n = \pm 1$. The first property is quite easy to prove but the second is deep and difficult. However, we will prove neither result here. The main result needed to prove it is as follows.

Theorem 2.1. Let C be a simple closed curve in $\mathbb{S}^2$ and suppose p, q lie in different components of $\mathbb{S}^2 \setminus C$. Then, the inclusion map

$$j : C \hookrightarrow \mathbb{S}^2 \setminus \{p, q\} \quad \text{induces an isomorphism of fundamental groups.}$$

This theorem is a special case of a rather deep theorem of Algebraic Topology, concerning with what is called the linking number of two disjoint subspaces of $\mathbb{S}^{m+n+1}$, one homeomorphic to an m-sphere and the other homeomorphic to an n-sphere. This is related to the Alexander duality theorem. The special case of our theorem is that of a 0-sphere (a two-point space) and a 1-sphere (a simple closed curve) in $\mathbb{S}^2$.

2.2. *The Cauchy Integral Formula*

One of the central theorems in the study of functions of a complex variable is the one concerning the Cauchy integral formula for analytic functions. For the classical version of this theorem, we need to assume two things, which are the

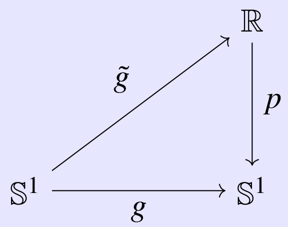
Jordan curve theorem and the winding number theorem (Theorem 2.1).

We begin our discussion with a more formal introduction of winding numbers.

Definition 2.2 (winding number). Let f be a loop in $\mathbb{R}^2$ and a be a point not in the image of f. Set

$$g(s) = \frac{f(s) - a}{\|f(s) - a\|} \quad \text{so } g \text{ is a loop in } \mathbb{S}^1.$$

Let $p : \mathbb{R} \to \mathbb{S}^1$ be the standard covering map and $\widetilde{g}$ be the lifting of g to $\mathbb{S}^1$. Since g is a loop, the difference $\widetilde{g}(1) - \widetilde{g}(0)$, which is an integer, is known as the winding number of f with respect to a, denoted by $n(f,a)$.



The lift $\widetilde{g}$ tracks how g winds around the circle. Since g is a loop, it makes sense that $\widetilde{g}(1)-\widetilde{g}(0)$ is an integer, which tells one how many times g winds around $\mathbb{S}^1$. Also, note that $n(f,a)$ is independent of the choice of the lifting of g, i.e. if $\widetilde{g}$ is one lifting of g, then uniqueness of lifting implies that any other lifting of g has the form $\widetilde{g}(s)+m$ for some $m \in \mathbb{Z}$.

Definition 2.3 (free homotopy). Let $F : I \times I \to X$ be a continuous map such that

$$F(0,t) = F(1,t) \quad \text{for all } t.$$

Then, for each t, the map $f_t(s) = F(s,t)$ is a loop in X. The map F is called a free homotopy between the loops f_0 and f_1. It is a homotopy of loops in which the base point of the loop is allowed to move during the homotopy.

Lemma 2.1. Let f be a loop in $\mathbb{R}^2 \setminus \{a\}$. Then, the following hold:

(a) If $\overline{f}$ is the reverse of f, then $n\left(\overline{f},a\right) = -n(f,a)$.

(b) If f is freely homotopic to f' through loops lying in $\mathbb{R}^2 \setminus \{a\}$, then $n(f,a) = n(f',a)$.

(c) If a and b lie in the same component of $\mathbb{R}^2 \setminus f(I)$, then $n(f,a) = n(f,b)$.

Proof. We first prove **(a)**. Recall that

$$n(f,a) = \widetilde{g}(1)-\widetilde{g}(0) \quad \text{where} \quad \widetilde{g}(s) \text{ is the lifting of } g(s) = \frac{f(s)-a}{\|f(s)-a\|} \text{ to } \mathbb{S}^1.$$

By replacing s with $1-s$, we can compute $n\left(\overline{f},a\right)$. Consequently, this changes $\widetilde{g}(1)-\widetilde{g}(0)$ by a sign.

To prove **(b)**, suppose F is a free homotopy between f and f'. Define

$$G : I \times I \to \mathbb{S}^1 \quad \text{by the equation} \quad G(s,t) = \frac{F(s,t)-a}{\|F(s,t)-a\|}.$$

Let $\widetilde{G}$ be a lifting of G to $\mathbb{R}$, so $\widetilde{G}(1,t)-\widetilde{G}(0,t)$ is an integer for all t. Since $\widetilde{G}$ is continuous, then it is a constant.

Lastly, we prove **(c)**. Let α be a path in $\mathbb{R}^2 \setminus f(I)$ from a to b. By definition of winding number, we have

$$n(f,a) = n(f-a,(0,0)).$$

This is obvious since n is invariant under translations, so the winding number of f around a is the same as the winding number of the translated map $f-a$ around the origin. Since $f(s)-\alpha(t)$ is a free homotopy in $\mathbb{R}^2 \setminus \{(0,0)\}$ between $f-a$ and $f-b$, the result follows. $\square$

Definition 2.4 (simple loop)**.** Let f be a loop in X. We say that f is a simple loop if f is injective, i.e. $f(s)=f(s')$ implies $s=s'$, or if one of the points s,s' is 0 and the other is 1.

Proposition 2.1. If $f: X \to X$ is a simple loop, then its image set is a simple closed curve in X.

Theorem 2.2. Let f be a simple loop in $\mathbb{R}^2$. We have

$$n(f,a)=\begin{cases} 0 & \text{if } a \text{ lies in the unbounded component of } \mathbb{R}^2 \setminus f(I); \\ \pm 1 & \text{otherwise.} \end{cases}$$

Definition 2.5 (counterclockwise and clockwise loops)**.** Let f be a simple loop in $\mathbb{R}^2$. f is a counterclockwise loop if $n(f,a)=1$ for some a (hence, for every a) in the bounded component of $\mathbb{R}^2 \setminus f(I)$. On the other hand, f is a clockwise loop if $n(f,a)=-1$.

Example 2.2. The standard loop $p(s)=(\cos 2\pi s, \sin 2\pi s)$ is a counterclockwise loop.

Theorem 2.3 (Cauchy integral formula)**.** Let C be a simple closed piecewise differentiable curve in the complex plane. Let B be the bounded component of $\mathbb{R}^2 \setminus C$. If $f(z)$ is analytic (can be expressed as a Taylor series) in an open set Ω that contains B and C, then for every $a \in B$, we have

$$f(a)=\frac{1}{2\pi i}\oint_C \frac{f(z)}{z-a}\,dz.$$

The sign is positive if C is oriented counterclockwise, and negative otherwise.

3. Group Theory

3.1. *Generation of Modules*

Recall from MA3201 the following: if we have an R-module M and let $\{N_\lambda\}_{\lambda\in\Lambda}$ be any collection of R-submodules of M, then

$$\bigcap_{\lambda\in\Lambda} N_\lambda \quad \text{is also an } R\text{-submodule of } M.$$

In other words, the arbitrary intersection of R-submodules of M is also an R-submodule of M. In fact, the intersection is the largest R-submodule contained in every N_λ.

Every Abelian group can be regarded as a $\mathbb{Z}$-module. So, we have the following fact:

$$\{\text{Abelian groups}\} \subseteq \{R\text{-modules}\}$$

Definition 3.1 (submodule generated by a subset)**.** Let M be an R-module. For every $A \subseteq M$, let

$$RA = \left\{\sum_{i=1}^{m} r_i a_i : r_i \in R, a_i \in A \text{ for all } 1 \leq i \leq m, m \in \mathbb{Z}^+\right\} \quad \text{be the submodule of } M \text{ generated by } A.$$

Alternatively, Definition 3.1 can be interpreted as follows. For any $A \subseteq M$, consider the collection of all R-submodules N of M containing A, i.e. $A \subseteq N$. Then, the intersection over this collection is thus also an R-submodule of M. It is called the R-submodule of M generated by A, denoted by RA or (A) (this shares a similar notion with ideals). As mentioned, this intersection is the smallest submodule of M which contains A.

For any $A \subseteq M$, we can also interpret RA as follows:

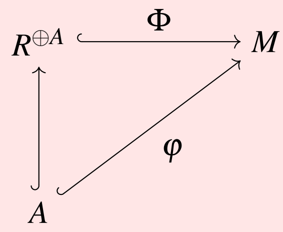
Proposition 3.1. *RA can be interpreted as the set of finite sums in M of the form $r_1\alpha_1 + \ldots + r_k\alpha_k$, where $r_1, \ldots, r_k \in R$ and $\alpha_1, \ldots, \alpha_k \in A$.*

If $A = \{a_1, \ldots, a_n\} \subseteq M$ is a finite set, we shall write

$$Ra_1 + \ldots + Ra_n \quad \text{to denote } RA, \quad \text{which is equivalent to} \quad \mathbb{Z}a_1 + \ldots + \mathbb{Z}a_n.$$

Theorem 3.1 (universal property of free modules)**.** The subset $A \subseteq M$ corresponds to the set-map $\varphi : A \hookrightarrow M$. By the universal property of the free R-module $R^{\oplus A}$, there exists a unique

R-module homomorphism such that the following diagram commutes:



Then, $RA = \Phi\left(R^{\oplus A}\right)$ is the image submodule of this homomorphism.

Definition 3.2 (generating set of a module). If N is a submodule of M (possibly $N = M$) and $N = RA$ for some $A \subseteq M$, we say that A is a set of generators or generating set for N, and we say N is generated by A.

This is equivalent to saying that N is generated by the subset $A \subseteq M$ if and only if the image module $\Phi\left(R^{\oplus A}\right)$ in M is equal to N.

Definition 3.3 (finitely generated module). A submodule N of M (possibly $N = M$) is finitely generated if there exists some finite subset $A \subseteq M$ such that $N = RA$, i.e. if N is generated by some finite subset.

Equivalently, there exists a finite subset $A \subseteq M$ such that N is equal to the image module $\Phi\left(R^{\oplus A}\right)$ in M if and only if there exists $n \in \mathbb{N}$ such that N is isomorphic to a quotient of the free module R^n of rank n.

Definition 3.4 (cyclic submodule). A submoule N of M (possibly $N = M$) is cyclic if there exists an element $a \in M$ such that $N = Ra$, that is, if N is generated by one element, i.e.

$$N = Ra = \{ra : r \in R\}.$$

Equivalently, there exists $a \in M$ such that N is equal to the image module $\Phi(R) = \{ra \in M : r \in R\}$ in M if and only if $N \cong R/I$ for some ideal I of R.

Definition 3.5. Let M be an R-module and let $\{N_\lambda\}_{\lambda \in \Lambda}$ be any collection of R-submodules of M. Take

$$A = \bigcup_{\lambda \in \Lambda} N_\lambda.$$

The sum of the R-submodules N_λ is equal to

$$\sum_{\lambda\in\Lambda} N_\lambda = \left\{\text{finite sums } x_{\lambda_1}+\ldots+x_{\lambda_k} : \lambda_1,\ldots,\lambda_k \in \Lambda \text{ and } x_{\lambda_i} \in N_{\lambda_i}\right\}.$$

It is the smallest submodule of M containing every N_λ.

Definition 3.6 (direct product)**.** Let M,N be R-modules. The direct product of M and N is the R-module $M\times N$ which can be regarded as the product of the additive groups $(M,+,0_M,-)$ and $(N,+,0_N,-)$ and the properties of scalar multiplication are satisfied.

In general, let $\{M_\alpha\}_{\alpha\in A}$ be a family of R-modules (indexed by A). The direct product of the family $\{M_\alpha\}_{\alpha\in A}$ is the R-module

$$\prod_{\alpha\in A} M_\alpha$$

with an underlying additive group and the properties of scalar multiplication are satisfied.

Lemma 3.1. The canonical projection maps

$$\pi_M : M\times N \to M \text{ where } (m,n)\mapsto m \quad\text{and}\quad \pi_N : M\times N\to N \text{ where } (m,n)\mapsto n$$

are surjective R-module homomorphisms with

$$\ker(\pi_M) = \{0_M\}\times N \quad\text{and}\quad \ker(\pi_N) = M\times\{0_N\}.$$

Definition 3.7 (direct sum)**.** Let M,N be R-modules. The direct sum of M and N is the R-module $M\times N$, denoted by $M\oplus N$.

In general, if $\{M_\alpha\}_{\alpha\in A}$ is a family of R-modules (indexed by A), the direct sum of the family $\{M_\alpha\}_{\alpha\in A}$ is the R-module

$$\bigoplus_{\alpha\in A} M_\alpha$$

which is defined to be the R-submodule of

$$\prod_{\alpha\in A} M_\alpha \quad \text{consisting of elements which are zero in almost all components.}$$

Example 3.1 (Munkres p. 411 Question 1)**.** Suppose that $G=\sum G_\alpha$. Show this sum is direct if and only if the equation

$$x_{\alpha_1}+\cdots+x_{\alpha_n} = 0$$

implies that each x_{α_i} equals 0. Here $x_{\alpha_i}\in G_{\alpha_i}$ and the indices α_i are distinct.

Solution. We first prove the forward direction. Suppose the sum is direct. Given the uniqueness of the representation and that $0 = 0 + \ldots + 0$ regardless of which groups are considered in the sum, it follows that $x_{\alpha_i} = 0$ for all $1 \le i \le n$.

We then prove the reverse direction. Suppose for some $x \in G$, we have two decompositions as follows:

$$x = x_{\alpha_1} + \ldots + x_{\alpha_n} = y_{\beta_1} + \ldots + y_{\beta_m}.$$

Then, $x_{\alpha_1} + \ldots + x_{\alpha_n} - y_{\beta_1} - \ldots - y_{\beta_m} = 0$. If there exist i and j such that $\alpha_i = \beta_j$, then $x_{\alpha_i} = y_{\beta_j}$, and for all other indices α_i and β_j, we must have $x_{\alpha_i} = y_{\beta_j} = 0$, implying that the sum is direct. □

Example 3.2 (Munkres p. 411 Question 2)**.** Show that if G_1 is a subgroup of G, there may be no subgroup G_2 of G such that $G = G_1 \oplus G_2$.
Hint: Set $G = \mathbb{Z}$ and $G_1 = 2\mathbb{Z}$.

Solution. Take $G = \mathbb{Z}$ and $G_1 = 2\mathbb{Z}$. One checks that $G_1 \le G$. Suppose on the contrary that there exists $G_2 \le G$ such that $G = G_1 \oplus G_2$. Then, G_2 must be the set of odd integers (since the sum $G_1 + G_2$ is direct and they must generate the group of integers $\mathbb{Z}$). However, the closure property in the set of odd integers $2\mathbb{Z} + 1$ fails since the sum of two odd integers is not odd. □

Lemma 3.2. The canonical inclusion maps

$$i_M : M \to M \oplus N \text{ where } m \mapsto (m, 0_N) \quad \text{and} \quad i_N : N \to M \oplus N \text{ where } n \mapsto (0_M, n)$$

are injective R-module homomorphism with

$$i_M(M) = M \times \{0_N\} \quad \text{and} \quad i_N(N) = \{0_M\} \times N.$$

3.2. *Free Modules*

Definition 3.8 (free module)**.** For any set A, the free R-module on the set A is the R-module

$$F(A) = R^{\oplus A} = R^{(A)} = \bigoplus_{\alpha \in A}.$$

In other words, the free R-module is the direct sum of the family of R-modules indexed by A, with each member being a copy of the regular R-module R.

Definition 3.9 (free module)**.** An R-module F is said to be free if and only if there exists a set A such that $F \cong F(A)$ as R-modules, i.e. there exists an R-module isomorphism $f : F \to F(A)$.

Lemma 3.3. If two sets A and B have the same cardinality, then $F(A) \cong F(B)$ as R-modules.

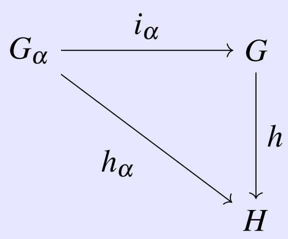
Proposition 3.2. Suppose $R \neq 0$. If two sets A and B are such that $F(A) \cong F(B)$ as R-modules, then they have the same cardinality.

Corollary 3.1 (rank of free module)**.** Suppose $R \neq 0$. If F is a free R-module, then the cardinality of A is well-defined and it is called the rank of F as a free R-module.

Example 3.3. For each $n \in \mathbb{N}$, the set $R^n = \{(r_1, \ldots, r_n) : r_i \in R \text{ for all } i\}$ is a free module of rank n over R.

3.3. *Free Products of Groups*

Definition 3.10 (free product and its universal property)**.** Let $\{G_\alpha\}_{\alpha \in J}$ be an indexed family of groups. The free product of the groups G_α is a group G given with a family $i_\alpha : G_\alpha \to G$ of homomorphism satisfying the following universal property: for any group and any family of homomorphism $h_\alpha : G_\alpha \to H$, there exists a unique homomorphism $h : G \to H$ such that for each $\alpha \in J$, the following diagram commutes:



Theorem 3.2 (existence of free products)**.** Let $\{G_\alpha\}_{\alpha \in J}$ be a family of groups. Then, there exists a free product of the groups G_α, i.e. there exists a group G with a family $i_\alpha : G_\alpha \to G$ of homomorphism satisfying the universal property (Definition 3.10).

Theorem 3.3 (uniqueness of free products)**.** Let $\{G_\alpha\}_{\alpha \in J}$ be a family of groups. Suppose

$$G \text{ with } i_\alpha : G_\alpha \to G \quad \text{and} \quad G' \text{ with } i'_\alpha : G_\alpha \to G'$$

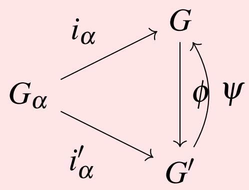
are both free products of the groups G_α in the sense of having the universal property (Definition 3.10). Then, there exist canonical isomorphism

$$\phi : G \to G' \quad \text{and} \quad \psi : G' \to G$$

such that for each $\alpha \in J$, we have

$$\phi \circ i_\alpha = i'_\alpha \quad \text{and} \quad \psi \circ i'_\alpha = i_\alpha.$$

In other words, the following diagram commutes:



Thus, the free product of groups G_α exists, and it is unique up to a canonical isomorphism. It is denoted by

$$G = \coprod_{\alpha \in J} G_\alpha = \mathop{\Huge *}_{\alpha \in J} G_\alpha \quad \text{or in the finite case,} \quad G = G_1 * \ldots * G_n.$$

Lemma 3.4. Let $\{G_\alpha\}_{\alpha \in J}$ be a family of groups. Suppose G with $i_\alpha : G_\alpha \to G$ is a free product of the groups G_α. Then, the following hold:

(a) each $i_\alpha : G_\alpha \to G$ is injective

(b) the family $\{i_\alpha(G_\alpha)\}_{\alpha \in J}$ of subgroups of G generate G.

Corollary 3.2. Let $G = G_1 * G_2$ be a free product of groups, where

$$G_1 \text{ is the free product of } \{H_\alpha\}_{\alpha \in J} \quad \text{and} \quad G_2 \text{ is the free product of } \{H_\beta\}_{\beta \in K}.$$

If $J \cap K = \emptyset$, then G is the free product of the subgroups $\{H_\gamma\}_{\gamma \in J \cup K}$. In particular,

$$G_1 * (G_2 * G_3) = G_1 * G_2 * G_3 = (G_1 * G_2) * G_3.$$

Definition 3.11 (least normal subgroup)**.** If S is a subset of a group G, the normal subgroup of G generated by S is the intersection of all normal subgroups of G that contain S. It is the least normal subgroup of G that contains S. Moreover, it is the subgroup of G generated by all conjugates of elements of S.

Theorem 3.4. Let $N_i \trianglelefteq G_i$ for $i = 1, 2$ and $N \trianglelefteq (G_1 * G_2)$ is generated by N_1 and N_2. Then,

$$(G_1 * G_2)/N \cong (G_1/N_1) * (G_2/N_2) \quad \text{canonically.}$$

3.4. *Free Groups*

Definition 3.12 (free group)**.** Let $\{a_\alpha\}$ be a family of elements of a group G. Suppose each a_α generates an infinite cyclic subgroup G_α of G. If G is the free product of the groups $\{G_\alpha\}$, then G is said to be a free group and the family $\{a_\alpha\}$ is a system of free generators for G.

Proposition 3.3. Let G be a group; let $\{a_\alpha\}_{\alpha\in J}$ be a family of elements of G. If G is a free group with system of free generators $\{a_\alpha\}$, then G satisfies the following condition:

$$\text{given any group } H \quad \text{and} \quad \text{any family } \{y_\alpha\} \text{ of elements of } H,$$
$$\text{there exists a homomorphism } h : G \to H \quad \text{such that} \quad h(a_\alpha) = y_\alpha \text{ for each } \alpha$$

Corollary 3.3. Let

$$G = G_1 * G_2 \quad \text{be the free product of two free groups.}$$

Here, G_1 and G_2 are free groups with $\{a_\alpha\}_{\alpha\in J}$ and $\{a_\alpha\}_{\alpha\in K}$ as respective systems of free generators. If $J \cap K = \emptyset$, then

$$G \text{ is a free group} \quad \text{with} \quad \{a_\alpha\}_{\alpha\in J\cup K} \text{ as a system of free generators.}$$

Example 3.4 (Munkres p. 412 Question 3)**.** If G is free abelian with basis $\{x,y\}$, show that $\{2x+3y, x-y\}$ is also a basis for G.

Solution. G being free Abelian simply means that it is Abelian with a basis. In particular, any element in G can be uniquely expressed as a linear combination of x and y with integer coefficients. Analogous to concepts in MA2001, it suffices to show that the matrix

$$\begin{bmatrix} 2 & 3 \\ 1 & -1 \end{bmatrix} \quad \text{is invertible,}$$

which is easy to verify. □

Example 3.5 (Munkres p. 412 Question 4)**.** The *order* of an element a of an abelian group G is the smallest positive integer m such that $ma = 0$, if such exists; otherwise, the order of a is said to be infinite. The order of a thus equals the order of the subgroup generated by a.

(a) Show the elements of finite order in G form a subgroup of G, called its *torsion subgroup*.

(b) Show that if G is free abelian, it has no elements of finite order.

(c) Show the additive group of rationals has no elements of finite order, but is not free abelian. *Hint*: If $\{a_\alpha\}$ is a basis, express $\frac{1}{2}a_\alpha$ in terms of this basis.

Solution.

(a) Let G be an Abelian group and $a_1, a_2 \in G$ be of finite order, i.e. there exist smallest $m_1, m_2 \in \mathbb{Z}^+$ such that $m_1a_1 = 0$ and $m_2a_2 = 0$. Then, $\operatorname{lcm}(m_1,m_2)(a_1+a_2) = 0$, satisfying the closure property. The other three group axioms are easy to verify.

(b) Suppose G is free Abelian. Then, G is Abelian with a basis. Let

$$a = \sum_{\alpha\in J} m_\alpha a_\alpha \quad \text{so given that} \quad ma = \sum_{\alpha\in J} m m_\alpha a_\alpha = 0,$$

we must have $mm_\alpha = 0$. This forces $mm_\alpha a = 0$ for all α. As such, $a = 0$. Since we are only interested in elements which are non-zero, it follows that G has no non-zero element of finite order.

(c) Consider $G = (\mathbb{Q}, +)$. Suppose on the contrary that there exists a non-zero element $x \in G$ of finite order, i.e. there exists $m \in Z^+$ such that $mx = 0$. Then, $m = 0$ or $x = 0$, which yields a contradiction.

Given that

$$\frac{1}{2}a_\beta = \sum_{\alpha \in J} m_\alpha a_\alpha \quad \text{this implies} \quad a_\beta = \sum_{\alpha \in J} 2m_\alpha a_\alpha.$$

As such, $2m_\beta = 1$, which is a contradiction. □

Example 3.6 (Munkres p. 412 Question 5)**.** Give an example of a free abelian group G of rank n having a subgroup H of rank n for which $H \neq G$.

Solution. Note that $\mathbb{Z}$ is free Abelian of rank 1 since it is generated by the element 1, i.e. $G = \langle 1 \rangle$. Consider $G = \mathbb{Z}$ and we note that $H = 2\mathbb{Z} \leq \mathbb{Z} = G$, where H is of rank 1 since H is generated by the element 2, i.e. $2\mathbb{Z} = \langle 2 \rangle$. However, $H \neq G$. □

Recall the following fact from MA2202.

Definition 3.13 (commutator)**.** For any $x, y \in G$, the commutator of x and y is the element

$$[x,y] = xyx^{-1}y^{-1} \quad \text{of } G.$$

Definition 3.14 (commutator subgroup)**.** The commutator subgroup of G is the subgroup of G generated by all commutators in G, i.e.

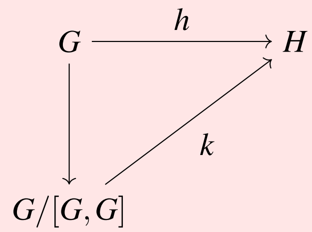
$$[G,G] = \langle [x,y] : x, y \in G \rangle.$$

Proposition 3.4 (commutator subgroup is normal subgroup)**.** $[G,G] \trianglelefteq G$

Lemma 3.5. $G/[G,G]$ is the maximal Abelian quotient of G with the following universal property: for any homomorphism

$$h : G \to H \quad \text{from } G \text{ to an Abelian group } H,$$

there exists a unique homomorphism $k : G/[G,G] \to H$ such that the following diagram commutes:



Theorem 3.5. If G is the free group on the set J, then $G/[G,G]$ is the free Abelian group on the set J.

Definition 3.15 (presentation of a group)**.** A presentation of a group G, denoted by $G = \langle S \mid R \rangle$, consists of

$$\text{a subset } S \subseteq G \text{ comprising the generators} \quad \text{and}$$
$$\text{a subset } R \subseteq \mathscr{F}(S) \text{ comprising a complete set of relations}$$

such that the canonical homomorphism $\mathscr{F}(S) \xrightarrow{h} G$ which maps $a \in S$ to $a \in G$ is surjective.

Definition 3.16 (finitely generated and finitely presented group)**.** The group G is finitely generated if and only if

$$\text{there exists a presentation } G = \langle S \mid R \rangle \quad \text{with } S \text{ finite.}$$

The group G is finitely presented if and only if

$$\text{there exists a presentation } G = \langle S \mid R \rangle \quad \text{with both } S \text{ and } R \text{ finite.}$$

Say G is finitely presented with

$$S = \{s_1, \ldots, s_n\} \quad \text{and} \quad R = \{w_1, \ldots, w_k\}.$$

Then, we write

$$G = \langle s_1, \ldots, s_n : w_1 = \ldots = w_k = 1 \rangle$$

and if w is the word $w_1 w_2^{-1}$, we write $w_1 = w_2$ instead of $w = 1$.

Example 3.7 (finite group has finite presentation)**.** Every finite group $G = \{g_1, \ldots, g_n\}$ is finitely presented. To see why, take $S = G$ and R to be the set of words $g_i g_j g_k^{-1}$ where $g_i g_j = g_k$ in G.

Example 3.8 (presentation of $\mathbb{Z}$)**.** We have $\mathbb{Z} \cong F(\{a\} = \langle a \rangle$. In fact, this is a trivial result that was covered in MA2202.

To see why this is true, no relations are imposed on a so a can be interpreted as the generator of

$\mathbb{Z}$, where powers of a correspond to integers. Specifically, a^n corresponds to the integer n and a^{-n} corresponds to the integer $-n$. Hence, $\mathbb{Z}$ is isomorphic to a group generated by a single element with no further relations.

Example 3.9 (presentation of $\mathbb{Z}\times\mathbb{Z}$). We have

$$\mathbb{Z}\times\mathbb{Z}\cong\langle a,b \mid [a,b]=1\rangle .$$

The only relation here is that a and b commute, i.e. $[a,b]=1$. This means that $ab=ba$. Note that $\mathbb{Z}\times\mathbb{Z}$ consists of ordered pairs of integers, and in terms of generators, each generator a and b corresponds to a copy of $\mathbb{Z}$, with the two copies commuting.

Example 3.10 (presentation of $\mathbb{Z}^n\times\mathbb{Z}^m$). We have

$$\mathbb{Z}_n\times\mathbb{Z}_m\cong\langle a^n=b^m=[a,b]=1\rangle .$$

Example 3.11 (presentation of quaternion group Q_8). Let $\mathbb{Q}_8$ be the quaternion group, which is a non-Abelian group of order 8, consisting of elements that are closely related to the quaternions. The quaternion group Q_8 is defined by the following set of elements:

$$Q_8=\{1,-1,i,-i,j,-j,k,-k\}$$

with multiplication rules given as follows:

(1). $i^2=j^2=k^2=-1$

(2). $ij=k$ and $ji=-k$

(3). $jk=i$ and $kj=-i$

(4). $ki=j$ and $ik=-j$

We note that

$$Q_8=\left\langle i,j \mid i^4=1, j^2=i^2, j^{-1}ij=i^{-1}\right\rangle .$$

Example 3.12 (presentation of dihedral group D_{2n}). Recall that the group D_{2n} is the dihedral group of order $2n$ and it represents the symmetries of a regular n-gon, including both rotations r and reflections s. Here, r represents a rotation by $2\pi/n$ in the anticlockwise direction; s represents a reflection symmetry.

We have

$$D_{2n}=\left\langle r,s \mid r^n=s^2=1, s^{-1}rs=r^{-1}\right\rangle .$$

Example 3.13 (presentation of symmetric group S_n). We have

$$S_n\cong\left\langle t_1,\ldots,t_{n-1} \mid t_i^2=1,(t_i,t_{i+1})^3=1,[t_i,t_j]=1 \text{ whenever } |i-j|\geq 2, 1\leq i,j\leq n-1\right\rangle .$$

There is an interesting problem in Group Theory known as the Group Isomorphism Problem, which states that given two finite presentations of groups, determine whether or not the groups are isomorphic. There is no effective procedure for solving this problem.

4. Classification of Covering Spaces

4.1. *Equivalence of Covering Spaces*

Throughout, unless stated otherwise, all topological spaces are assumed to be path-connected and locally path-connected. Let B be a topological space, i.e. a base space of interest. Recall that a B-space is a topological space E equipped with a continuous map $p : E \to B$.

Definition 4.1 (equivalence of covering maps/covering spaces)**.** Let

$$p : E \to B \quad \text{and} \quad p' : E' \to B \quad \text{be covering spaces of } B.$$

An isomorphism of B-spaces $h : E \to E'$ is an equivalence of covering maps/covering spaces. If there exists such an equivalence, then p and p' are equivalent as covering maps of B (alternatively, E and E' are equivalent as covering spaces of B).

Theorem 4.1. Let $p : E \to B$ and $p' : E' \to B$ be covering spaces of B and let $b_0 \in B$. Let $e_0 \in p^{-1}(b_0)$ and $e'_0 \in (p')^{-1}(b_0)$ be in the fiber over b_0. Then, the following hold:

(i) There exists an equivalence $h : E \to E'$ of covering spaces with $h(e_0) = e'_0$ if and only if

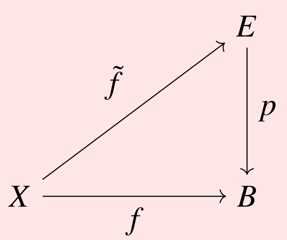
$$p_*(\pi_1(E, e_0)) \text{ and } p'_*(\pi_1(E', e'_0)) \quad \text{are equal as subgroups of } \pi_1(B, b_0).$$

If such an equivalence h exists, it is unique.

(ii) There exists an equivalence $h : E \to E'$ of covering spaces if and only if

$$p_*(\pi_1(E, e_0)) \text{ and } p'_*(\pi_1(E', e'_0)) \quad \text{are conjugate as subgroups of } \pi_1(B, b_0).$$

Lemma 4.1 (the general lifting lemma)**.** Let $p : E \to B$ be a covering map and let $b_0 \in B$. Let $e_0 \in p^{-1}(b_0)$ be in the fiber over b_0. Suppose Y is path-connected and locally path-connected. Let $f : Y \to B$ be a continuous map and let $y_0 \in Y$ be such that $f(y_0) = b_0$. Then, there exists a lifting $\widetilde{f} : Y \to E$ of f with $\widetilde{f}(y_0) = e_0$ if and only if $f_*(\pi_1(Y, y_0)) \subseteq p_*(\pi_1(E, e_0))$ as subgroups of $\pi_1(B, b_0)$. If such a lifting exists, it is unique.



Corollary 4.1. The map

$$\begin{aligned}&\{\text{equivalence classes of path-connected covering spaces of } B\}\\ \to &\{\text{conjugacy classes of subgroups of } \pi_1(B,b_0)\}\end{aligned}$$

given by

$$[p:E\to B]\mapsto [p_*(\pi_1(E,e_0)) \quad \text{for any choice of } e_0\in p^{-1}(b_0)] \quad \text{is well-defined and injective.}$$

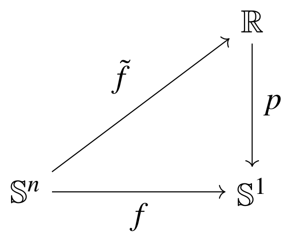
Remark 4.1. If $\pi_1(B,b_0)$ is Abelian, then

$$\{\text{conjugacy classes of subgroups of } \pi_1(B,b_0)\} = \{\text{subgroups of } \pi_1(B,b_0)\}.$$

Example 4.1 (Munkres p. 483 Question 1)**.** Use the lifting lemma to show that for $n>1$,

$$\text{every continuous map } f:\mathbb{S}^n\to\mathbb{S}^1 \quad \text{is nulhomotopic.}$$

Solution. Recall that if f and f' are paths such that $f\sim f'$ and f' is the constant map, then f is said to be nulhomotopic. Consider the following commutative diagram:



Suppose

$$p:\mathbb{R}\to\mathbb{S}^1 \quad \text{defined by} \quad p(x)=e^{2\pi ix} \quad \text{is a covering map.}$$

As a side remark, one should already perceive the above-mentioned *trick* as second nature at this point. Since $\pi_1(\mathbb{S}^n)$ is the trivial group for $n>2$, then for any continuous map $f:\mathbb{S}^n\to\mathbb{S}^1$, we see that $f_*:\pi_1(\mathbb{S}^n)\to\pi_1(\mathbb{S}^1)$ is trivial, so it is a subgroup of $\pi_1(\mathbb{S}^1)$.

As such, the lifting $\widetilde{f}:\mathbb{S}^n\to\mathbb{R}$ is a map from a compact space to $\mathbb{R}$, so $\widetilde{f}$ is homotopic to a constant map. Since $f=p\circ\widetilde{f}$, it follows that f is also homotopic to a constant map. □

Example 4.2. Recall the monodromy action of the exponential covering map, which refers to how the fibers of the covering space transform as we go around loops in the base space. We have

$$\pi_1(\mathbb{S}^1,1)\to\mathbb{Z} \text{ (in fact it is an isomorphism)} \quad \text{where} \quad \alpha\mapsto 0*\alpha$$

via the monodromy action of the exponential covering map $p:\mathbb{R}\to\mathbb{S}^1$, where $x\mapsto e^{2\pi ix}$. Since $\mathbb{R}$ is simply-connected, then $p_*(\pi_1(\mathbb{R},0))=0$ in $\pi_1(\mathbb{S}^1,1)\cong\mathbb{Z}$.

By Corollary 4.1, the map

$$\{\text{equivalence classes of path-connected covering spaces of } \mathbb{S}^1\} \to \{\text{subgroups of } \pi_1(\mathbb{S}^1,1) \cong \mathbb{Z}\}$$

given by

$$[p' : E' \to \mathbb{S}^1] \mapsto [p'_*(\pi_1(E', e'_0)) \quad \text{for any choice of } e'_0 \in (p')^{-1}(1)] \quad \text{is well-defined and injective.}$$

Example 4.3. Also, as another example, for any non-zero integer n,

$$\text{the } n^{\text{th}} \text{ power map} \quad p_n : \mathbb{S}^1 \to \mathbb{S}^1 \text{ with } z \mapsto z^n \quad \text{is a covering map.}$$

Then, $(p_n)_*(\pi_1(\mathbb{S}^1,1)) = n \cdot \pi_1(\mathbb{S}^1,1) \cong n\mathbb{Z}$.

Recall the following basic fact from Group Theory, which states that the map $\mathbb{Z}_{\geq 0} \to \{\text{subgroups of } \mathbb{Z}\}$ is bijective, where $n \mapsto n\mathbb{Z}$. Applying Corollary 4.1, we see that the map

$$\{\text{equivalence classes of path-connected covering spaces of } \mathbb{S}^1\} \to \{\text{subgroups of } \pi_1(\mathbb{S}^1,1) \cong \mathbb{Z}\}$$

is bijective. Moreover, the equivalence classes of path-connected covering spaces of $\mathbb{S}^1$ is precisely the exponential map $p : \mathbb{R} \to \mathbb{S}^1$, which is also equal to the set of n^{th} power maps $p_n : \mathbb{S}^1 \to \mathbb{S}^1$ with $n \in \mathbb{Z}_{>0}$.

In particular, for any $n \in \mathbb{Z}_{>0}$, p_n and p_{-n} are equivalent as covering maps (this is closely related to Example 1.25 from one of the exercises in Munkres' textbook), i.e. the following diagram commutes:

$$\begin{array}{ccccc} z & & \longmapsto & & z^{-1} \\ & & \cong & & \\ \mathbb{S}^1 & & \longrightarrow & & \mathbb{S}^1 \\ & p_n \searrow & & \swarrow p_{-n} & \\ & & \mathbb{S}^1 & & \end{array}$$

Example 4.4 (Munkres p. 483 Question 2)**.**

(a) Show that every continuous map $f : \mathbb{P}^2 \to \mathbb{S}^1$ is nullhomotopic.

(b) Find a continuous map of the torus into $\mathbb{S}^1$ that is not nullhomotopic.

Solution.

(a) The idea here is to use the fundamental groups of $\mathbb{P}^2$ and $\mathbb{S}^1$. The fundamental group of the real projective plane $\mathbb{P}^2$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$, whereas the fundamental group of the circle $\mathbb{S}^1$ is isomorphic to $\mathbb{Z}$. So,

$$\text{for any continuous map } f : \mathbb{P}^2 \to \mathbb{S}^1 \quad \text{it induces a homomorphism on fundamental groups} \quad f_* : \pi_1\left(\mathbb{P}^2\right) \to$$

So, any homomorphism from $\mathbb{Z}/2\mathbb{Z}$ to $\mathbb{Z}$ must map the entire group $\mathbb{Z}/2\mathbb{Z}$ to the identity in $\mathbb{Z}$ as there are no non-trivial homomorphisms from a finite group to an infinite group. As such, the induced map on the fundamental group is trivial, so any map f can be continuously deformed to a constant map.

(b) We wish to find

$$\text{a continuous map } f:\mathbb{T}\to\mathbb{S}^1 \quad \text{which is not nulhomotopic.}$$

Note that $\mathbb{T}=\mathbb{S}^1\times\mathbb{S}^1$. Say we define a projection map

$$p:\mathbb{T}\to\mathbb{S}^1 \quad \text{where} \quad p(x,y)=x.$$

By definition of the projection map, p is continuous. However, the homomorphism from the fundamental group of the torus to the fundamental group of the circle is surjective, and hence, non-trivial. As such, π_1 is not nulhomotopic. □

Example 4.5 (Munkres p. 483 Question 3)**.** Let $p:E\to B$ be a covering map; let $p(e_0)=b_0$. Show that $H_0=p_*(\pi_1(E,e_0))$ is a normal subgroup of $\pi_1(B,b_0)$ if and only if for every pair of points e_1,e_2 of $p^{-1}(b_0)$, there is an equivalence $h:E\to E$ with $h(e_1)=e_2$.

Solution. We first prove the forward direction. Suppose

$$p_*(\pi_1(E,e_0))\trianglelefteq\pi_1(B,b_0).$$

Take

$$\text{any loop } \gamma\in\pi_1(B,b_0) \quad \text{and} \quad \text{any element } \alpha\in\pi_1(E,e_0).$$

Then, the conjugate $\gamma\alpha\gamma^{-1}$ is also contained in $p_*(\pi_1(E,e_0))$. As such, any loop in B can be lifted to a path in E that maps one point in $p^{-1}(b_0)$ to another.

Take $e_1,e_2\in p^{-1}(b_0)$. By the covering space property, for any path γ in B from b_0 to itself (essentially a loop), each point in the fiber $p^{-1}(b_0)$ lifts this path uniquely to a path in E starting at that point in $p^{-1}(b_0)$. Since p_* is normal in $\pi_1(B,b_0)$, for any $e_1,e_2\in p^{-1}(b_0)$, we can construct a homeomorphism $h:E\to E$ by lifting appropriate loops in B. This homeomorphism can be chosen to map e_1 to e_2.

Proof of reverse direction is omitted. □

Example 4.6 (Munkres p. 483 Question 4)**.** Let $\mathbb{T}=\mathbb{S}^1\times\mathbb{S}^1$, the torus. There is an isomorphism of $\pi_1(T,b_0\times b_0)$ with $\mathbb{Z}\times\mathbb{Z}$ induced by projections of $\mathbb{T}$ onto its two factors.

(a) Find a covering space of $\mathbb{T}$ corresponding to the subgroup of $\mathbb{Z}\times\mathbb{Z}$ generated by the element $m\times 0$, where m is a positive integer.

(b) Find a covering space of $\mathbb{T}$ corresponding to the trivial subgroup of $\mathbb{Z}\times\mathbb{Z}$.

(c) Find a covering space of $\mathbb{T}$ corresponding to the subgroup of $\mathbb{Z}\times\mathbb{Z}$ generated by $m\times 0$ and $0\times n$, where m and n are positive integers.

Solution.

(a) The subgroup generated by $(m,0)$ is

$$\Gamma=\{(km,0):k\in\mathbb{Z}\}\cong m\mathbb{Z}\times\{0\}.$$

Since Γ acts by translations along the x-axis with period m, the quotient space $\mathbb{R}^2/\Gamma$ identifies points that differ by multiples of $(m,0)$, i.e. along the x-direction, the space is wrapped every m units; along the y-direction, the space remains unwrapped. So,

$$\mathbb{R}^2/\Gamma \quad \text{is homeomorphic to a cylinder.}$$

This implies that the covering map $p : \mathbb{R}^2/\Gamma \to \mathbb{T}$ is induced by the inclusion $\Gamma \subseteq \mathbb{Z}^2$. Explicitly, what the covering action does is that the cylinder wraps around the torus m times in the x-direction and infinitely in the y-direction. Hence,

$$p : \mathbb{R}^2/\Gamma \quad \text{where} \quad p([x,y]) = ([x \pmod 1], [y \pmod 1]).$$

(b) Since the subgroup Γ is trivial, then $\mathbb{R}^2/\Gamma = \mathbb{R}^2$. So, the covering space is simply the universal cover of $\mathbb{T}$, which is $\mathbb{R}^2$.

As such, the covering map is given as follows:

$$p : \mathbb{R}^2 \to \mathbb{T} \quad \text{where} \quad p([x,y]) = ([x \pmod 1], [y \pmod 1]).$$

(c) Let Γ denote the subgroup. Then, Γ is generated by $(m,0)$ and $(0,n)$, so

$$\Gamma = m\mathbb{Z} \times n\mathbb{Z}.$$

Since Γ is a lattice generated by $(m,0)$ and $(0,n)$, the covering space $\mathbb{R}^2/\Gamma$ is a torus, where the respective periods along the x- and y-directions are m and y.

Again, the covering map is the same as the preceding subparts, but in order to visualise the covering space, it is in fact a torus that covers T m times in one direction and n times in the other. □

4.2. *The Universal Covering Space*

By convention, throughout unless otherwise stated, all topological spaces are assumed to be

$$\text{path-connected} \quad \text{and} \quad \text{locally path-connected.}$$

Recall from MA3209 (I assume they cover this definition with no pun intended) that a locally path-connected topological space X is defined as follows: for all $x \in X$ and for all open neighbourhoods $V \subseteq X$ of x, there exists an open neighbourhood $U \subseteq V$ of x such that U is path-connected.

Definition 4.2 (universal covering space)**.** Let B be a topological space, i.e. a base space of interest. A universal covering space of B is a covering space $p : E \to B$ such that E is simply-connected.

Remark 4.2. There are examples of topological spaces B that have no universal covering space. Take the *infinite earring*/Hawaiian earring for instance.

Theorem 4.2 (universal property of a universal covering space)**.** Let $p : E \to B$ be a universal covering space of B. Suppose

$$b_0 \in B \quad \text{and} \quad e_0 \in p^{-1}(b_0) \text{ be in the fiber over } b_0.$$

Then, for any covering space $p' : E' \to B$ and any $e_0' \in (p')^{-1}(b_0)$,

$$\text{there exists a unique } B\text{-space map } h : E \to E' \quad \text{such that} \quad h(e_0) = e_0'.$$

Proof. Since E is simply-connected, its fundamental group is trivial, so

$$p_*(\pi_1(E, e_0)) = 1 \subseteq p_*'(\pi_1(E', e_0')) \quad \text{as subgroups of } \pi_1(B, b_0).$$

We then apply the general lifting lemma to the covering map $p' : E' \to B$ and the continuous map $p : E \to B$ to obtain this universal property. □

Lemma 4.2. Let

$$p : E \to B \quad \text{and} \quad p' : E' \to B \quad \text{be covering maps of } B.$$

Then, any B-space map $h : E \to E'$ is also a covering map.

We will not discuss the proof in detail. Briefly, construct a commutative diagram which involves $p' \circ h = p$. Then, one needs to show that for every point $x \in E$, there exists an open neighbourhood $V \subseteq E'$ of x which is evenly covered by h. In other words, one needs to show that

$$h \text{ is surjective} \quad \text{and} \quad E' \text{ is locally evenly covered by } h.$$

Lemma 4.3 (Munkres p. 487 Question 1)**.** Let

$$q : X \to Y \quad \text{and} \quad r : Y \to Z \quad \text{be maps} \quad \text{and} \quad p = r \circ q.$$

If q and r are covering maps and Z has a universal covering space, prove that p is a covering map.

Proof. Since Z has a universal covering space, say $\widetilde{Z}$, then there exists a covering map $\pi : \widetilde{Z} \to Z$ such that $\widetilde{Z}$ is simply-connected. We wish to prove that $p = r \circ q$ is a covering map, i.e. for every $z \in Z$, there exists an open neighbourhood $V \subseteq Z$ such that $p^{-1}(V)$ is a disjoint union of open sets in X, each of which is mapped homeomorphically onto V.

Since $r : Y \to Z$ is a covering map, then for every $z \in Z$, there exists an open set $V_z \subseteq Z$ such that

$$r^{-1}(V_z) = \bigsqcup_{i \in I_z} U_i \quad \text{such that} \quad r|_{U_i} : U_i \to V_z \text{ is homeomorphic.}$$

Similarly, for every $y \in r^{-1}(V_z)$, as $q : X \to Y$ is a covering map, then there exists an open set $W_y \subseteq Y$ such that

$$q^{-1}(W_y) = \bigsqcup_{j \in J_y} U_j \quad \text{such that} \quad q|_{U_j} : U_j \to W_y \text{ is homeomorphic.}$$

As $p^{-1}(V_z) = q^{-1}\left(r^{-1}(V_z)\right)$, we see that this is a disjoint union of open sets in X, each of which maps homeomorphically onto V_z under p. □

However, the composition of covering maps is not necessarily a covering map. As pointed out in the same exercise in Munkres, we need Z to be a universal covering space for $p : X \to Z$ to be a covering map. As an example of a topological space that does not have a universal covering space, consider the Hawaiian earring (will be discussed in due course).

Definition 4.3 (semi-locally simply-connected space). A space B is semi-locally simply-connected if and only if for any $b \in B$, there exists an open neighbourhood $U \subseteq B$ of b such that

$$\text{the homomorphism } i_* : \pi_1(U, b) \to \pi_1(B, b) \quad \text{induced by the inclusion } i : U \hookrightarrow B \quad \text{is trivial.}$$

In other words, any loop in U at b is nulhomotopic in B.

Remark 4.3. If $U \subseteq B$ is a semi-locally simply-connected space, then so is any smaller neighbourhood of b, i.e. we can shrink U if necessary.

Example 4.7 (simply-connected implies semi-locally simply-connected). Let B be simply-connected. Then, it is semi-locally simply-connected.

We provide some examples of the aforementioned result.

Example 4.8. Note that convex subsets of $\mathbb{R}^n$ are simply-connected. Hence, they are semi-locally simply-connected. To see why, consider a point $b \in B$, where $B \subseteq \mathbb{R}^n$. Then, there exists an open neighbourhood U of b such that any loop in U based at b is nulhomotopic in B. Here, we have exploited the convexity of subsets of $\mathbb{R}^n$.

Hence, $i_* : \pi_1(U, b) \to \pi_1(B, b)$ induced by the inclusion map $i : U \hookrightarrow B$ is trivial.

Example 4.9. Spheres $\mathbb{S}^n$, where $n > 1$ are simply-connected. Hence, they are semi-locally simply-connected.

Example 4.10. Let B be a locally simply-connected space, i.e. for all $b_0 \in B$ and open neighbourhoods $V \subseteq B$ of b_0, there exists an open neighbourhood $U \subseteq V$ of b_0 such that U is simply-connected.

Take for example,

$$B = \text{an arbitrary topological manifold.}$$

Then, B is semi-locally simply-connected.

Theorem 4.3. Let B be a path-connected and locally-path connected space, with $b_0 \in B$. Then, B is semi-locally simply-connected if and only if for any $H \leq \pi_1(B, b_0)$, there exists a covering space $p : E \to B$ and $e_0 \in p^{-1}(b_0)$ such that $H = p_*(\pi_1(E, e_0))$ as subgroups of $\pi_1(B, b_0)$.

Corollary 4.2. Let B be a path-connected and locally path-connected space. Then, there exists a universal covering space of B if and only if B is semi-locally simply-connected.

Example 4.11 (infinite earring/Hawaiian earring). Let C_n be the circle of radius $1/n$ in $\mathbb{R}^2$ with centre at the point $(1/n, 0)$. Let X be the union of these circles as a subspace of $\mathbb{R}^2$ (Figure 30), i.e.

$$X = \bigcup_{n=1}^{\infty} \left\{ (x,y) \in \mathbb{R}^2 : \left(x - \frac{1}{n} \right)^2 + y^2 = \left(\frac{1}{n} \right)^2 \right\}.$$

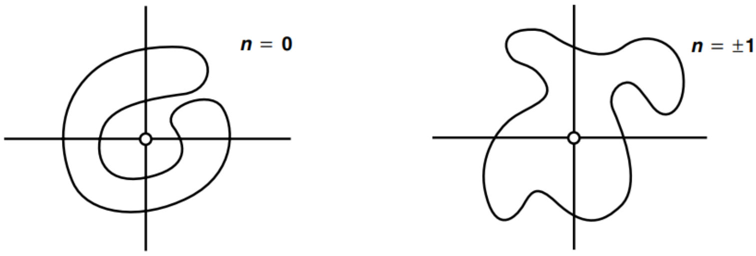


Figure 14: Winding numbers of some simple closed curves in $\mathbb{R}^2$

Then, X is path-connected and locally path-connected, but not semi-locally simply-connected. As such, X does not have a universal covering space. More precisely, if U is any neighbourhood of $b_0 = 0$, then

$$i_* : \pi_1(U, b_0) \to \pi_1(X, b_0) \quad \text{is non-trivial.}$$

Given n, there exists a retraction $r : X \to C_n$ obtained by letting r map each circle C_i to the point b_0, where $i \neq n$. Then, for n sufficiently large such that C_n lies in U, we see that in the following diagram homomorphism induced by inclusion, j_* is injective. Hence, i_* is non-trivial.

$$\begin{array}{ccc} \pi_1(C_n, b_0) & \xhookrightarrow{\;j_*\;} & \pi_1(X, b_0) \\ & {\scriptstyle k_*} \searrow \quad \swarrow {\scriptstyle i_*} & \\ & \pi_1(U, b_0) & \end{array}$$

Theorem 4.4. Let B be a path-connected and locally path-connected semilocally simply-connected space. Suppose $b_0 \in B$. Then, the map from

$$\{\text{equivalence classes of path-ctd. covering spaces of } B\} \to \{\text{conjugacy classes of subgroups of } \pi_1(B,b_0)\}$$

sending $[p : E \to B]$ to the equivalence class of $p_*(\pi_1(E,e_0))$ for any choice of $e_0 \in p^{-1}(b_0)$ is a well-defined order-reversing bijection.

We compare the above theorem with the Galois correspondence. Briefly recall the definition of $\mathrm{Gal}(L/K)$ from MA4203.

Theorem 4.5 (The Galois correspondence). Let k be a field and k^s be a separable algebraic closure of k. The map sending the set of k-isomorphism classes of algebraic separable extensions of k to conjugacy classes of subgroups of $\mathrm{Gal}(k^s/k)$, where

$$[k \hookrightarrow L] \mapsto [\mathrm{Gal}(k^s/i(L)) \text{ for any choice of } k\text{-embedding } i : L \hookrightarrow k^s]$$

is a well-defined order-reversing bijection.

4.3. *Covering Transformations*

Definition 4.4 (covering transformation/deck transformation). Let $p : E \to B$ be a covering space of B. A covering transformation, also known as a deck transformation, is an equivalence of $p : E \to B$ with itself, i.e. an isomorphism of B-spaces.

Definition 4.5 (covering transformation action). Let $\mathscr{C}(E,p,B)$ be the group of all covering transformations of p. This is in fact equal to

$$\mathrm{Aut}(E/B) = \mathrm{Aut}(P) \quad \text{which acts from the left on the space } E.$$

Let $b_{\in}B$. Then, $\mathrm{Aut}(p)$ acts from the left on the fiber $p^{-1}(b_0)$ over b_0, i.e.

$$\mathrm{Aut}(p) \times p^{-1}(b_0) \to p^{-1}(b_0) \quad \text{given by} \quad (h,e_0) \mapsto h(e_0) \quad \text{is the covering transformation action.}$$

Lemma 4.4. Let $p : E \to B$ be a covering map, with $b_0 \in B$ and $e_0 \in p^{-1}(b_0)$. The covering transformation $\mathrm{Aut}(p)$ on $p^{-1}(b_0)$ is fixed-point free, i.e.

$$\psi : \mathrm{Aut}(p) \to p^{-1}(b_0) \quad \text{such that} \quad h \mapsto h(e_0) \quad \text{is injective.}$$

Let $H = p_*(\pi_1(E,e_0)) \subseteq \pi_1(B,b_0)$. Define

$$N(H) = \{g : gHg^{-1} = H\} \quad \text{to be the normaliser of } H \text{ in } \pi_1(B,b_0).$$

Then, the ψ-image of $\mathrm{Aut}(p)$ in $p^{-1}(b_0)$ is equal to the Φ-image of $H/N(H)$ in $p^{-1}(b_0)$.

Proof. Since $h \in \text{Aut}(p)$ is a lifting of $p: E \to B$ over itself, by the general lifting lemma, h is uniquely determined by $h(e_0)$. As such, $\psi: \text{Aut}(p) \to p^{-1}(b_0)$ is injective.

Now, we need to prove that

$$\text{the } \psi\text{-image of } \text{Aut}(p) \text{ in } p^{-1}(b_0) = \Phi\text{-image of } H/N(H) \text{ in } p^{-1}(b_0).$$

We first prove the forward inclusion $\subseteq$. Suppose $h \in \text{Aut}(p)$ and $e_1 = h(e_0)$ in $p^{-1}(b_0)$. Then, $h^{-1}(e_1)$, so by the general lifting lemma, we have $p_*(\pi_1(E, e_0)) = p_*(\pi_1(E, e_1))$. Choose a path γ in E from e_0 to e_1. We can do so since E is path-connected. Also, let $\alpha = p \circ \gamma$ and a loop in B at b_0. Then,

$$[\alpha] * p_*(\pi_1(E, e_1)) * [\alpha]^{-1} = p_*(\pi_1(E, e_0)). \quad \text{as subgroups of } \pi_1(B, b_0).$$

Then,

$$[\alpha] \in N(H) \quad \text{in } \pi_1(B, b_0),$$

so

$$\Psi(h) = h(e_0) = e_1 = \gamma(1) = \Phi(H[\alpha]) \quad \text{lies in the } \Phi\text{-image of } H/N(H).$$

We briefly explain a couple of the equality signs. Note that $e_1 = \gamma(1)$ because of the choice of γ — it is a path in E from e_0 to e_1. As for $\gamma(1) = \Phi(H[\alpha])$, it follows by the monodromy action. We have thus proven $\subseteq$.

We then prove the reverse inclusion $\supseteq$. Suppose $[\alpha] \in N(H)$, represented by a loop α in B based at b_0. Let γ be the unique path lifting of α over p beginning at e_0. Let $e_1 = \gamma(1)$. Then, one has

$$[\alpha] * p_*(\pi_1(E, e_1)) * [\alpha]^{-1} = p_*(\pi_1(E, e_0)) \quad \text{since } [\alpha] \in N(H).$$

By the general lifting lemma, there exists a B-space map $h: E \to E$ with $h(e_0) = e_1$. By reversing roles, one sees that $h \in \text{Aut}(p)$ is invertible. Then,

$$\Phi(H[\alpha]) = \gamma(1) = e_1 = h(e_0) = \Psi(h) \quad \text{lies in the } \Psi\text{-image of } \text{Aut}(p).$$

The first equality follows by the definition of monodromy action. Also, note that the proof of the reverse inclusion mimics that of the forward inclusion. Anyway, the proof is complete. □

Theorem 4.6. Let $p: E \to B$ be a covering map, $b_0 \in B$ and $e_0 \in p^{-1}(b_0)$. Also, let

$$H = p_*(\pi_1(E, e_0)) \subseteq \pi_1(B, b_0).$$

The composite map

$$\Phi^{-1} \circ \psi : \mathrm{Aut}(p) \xrightarrow{\Psi} \Psi\text{-image in } p^{-1}(b_0) \xrightarrow{\Phi^{-1}} N(H)/H$$

is a group homomorphism, hence isomorphism.

Proof. Let $h, k \in \mathrm{Aut}(p)$ and let

$$e_1 = h(e_0) = \Psi(h) \quad e_2 = k(e_0) = \Psi(k) \quad e_3 = h(e_2) = h(k(e_0)) \quad \text{in } p^{-1}(b_0).$$

Choose paths γ, δ in E from e_0 to e_1, e_2 respectively. Let $\alpha = p \circ \gamma$ and $\beta = p \circ \delta$ be loops in B based at b_0, so

$$H[\alpha] = \Phi^{-1}(e_1), H[\beta] = \Phi^{-1}(e_2), H[\alpha * \beta] \quad \text{lie in } N(H)/H.$$

The path $h \circ \delta$ in E is from $h(e_0) = e_1$ to $h(e_2) = h(k(e_0)) = e_3$ and it is the unique path lifting of β over p beginning at e_1. Hence, $\gamma * (h \circ \delta)$ is the unique path lifting of $\alpha * \beta$ over p beginning at e_0, so it follows that

$$\Phi(H[\alpha * \beta]) = (\gamma * (h \circ \delta))(1) = e_3 = h(k(e_0)) = \Psi(h \circ k).$$

We briefly explain why $(\gamma * (h \circ \delta))(1) = e_3$. Recall that γ is a path from e_0 to $e_1 = h(e_0)$, $h \in \mathrm{Aut}(p)$ and δ is a path from e_1 to $e_2 = k(e_0)$. As such, the concatenation of paths γ with $h \circ \gamma$, denoted by $\gamma * (h \circ \delta)$, involves first considering a path that takes from e_0 to e_1 (by γ), then e_1 to e_3 (by $h \circ \delta$). Evaluation of the concatenation of paths at time $t = 1$ implies we should obtain e_3. □

Corollary 4.3. Let $p : E \to B$ be a universal covering space of B. Then, for any choice of $b_0 \in B$, $e_0 \in p^{-1}(b_0)$, the map

$$\mathrm{Aut}(p) \to \pi_1(B, b_0) \quad \text{given by} \quad h \mapsto [pi \circ \gamma]$$

for any path γ in B from e_0 to $h(e_0)$ is an isomorphism.

Definition 4.6 (regular/Galois covering map). Let $p : E \to B$ be a covering map. It is regular/Galois if and only if E is path-connected and $H = p_*(\pi_1(E, e_0)) \trianglelefteq \pi_1(B, b_0)$ for any choice of $b_0 \in B$ and $e_0 \in p^{-1}(b_0)$.

Corollary 4.4. A covering map p is Galois

if and only if $N(H) = \pi_1(B, b_0)$

if and only if the Ψ-image of $\mathrm{Aut}(p)$ in $p^{-1}(b_0)$ is $p^{-1}(b_0)$

Example 4.12. Since $\pi_1(\mathbb{S}^1, 1) \to \mathbb{Z}$ is Abelian (and isomorphic), then every covering of $\mathbb{S}^1$ is Galois.

To see why, recall that the set of equivalence classes of path-connected covering spaces of $\mathbb{S}^1$ is equivalent to the exponential map $p : \mathbb{R} \to \mathbb{S}^1$ given by $x \mapsto e^{2\pi i x}$. Recall that $\mathbb{R}$ is contractible so $\text{Aut}(p) \to \pi_1(\mathbb{S}^1, 1) \to \mathbb{Z}$ are isomorphic, where $\text{Aut}(p) \to \mathbb{Z}$ is given by $h \mapsto h(0)$.

Example 4.13. Consider the covering map

$$p_n : \mathbb{S}^1 \to \mathbb{S}^1 \quad \text{given by} \quad z \mapsto z^n, \quad \text{where } n \in \mathbb{Z}_{>0}.$$

We have $\text{Aut}(p_n) \to \pi_1(\mathbb{S}^1, 1)/p_{n_*}\pi_1(\mathbb{S}^1, 1) \to \mathbb{Z}/n\mathbb{Z}$ being isomorphic. Here, $h \mapsto \alpha \in \mathbb{Z}/n\mathbb{Z}$, where α is unique, such that $h(1) = e^{2\pi i/n}$, which is an n^{th} root of unity.

There is another example here. Remember to take from Lou Yi. Not sure if it is related to the fundamental group of the figure eight.

Definition 4.7 (set of homeomorphisms)**.** Let X be a topological space. Define

$$\text{Homeo}(X) \text{ to be the set of homeomorphisms of } X.$$

Definition 4.8 (G-action)**.** Let

$$X \text{ be a set} \quad \text{and} \quad G \leq \text{Aut}(X) = \text{Perm}(X).$$

The orbit space X/G is the quotient set of X modulo the equivalence relation $\sim$ of G-action, i.e.

$$x_1 \sim x_2 \quad \text{if and only if} \quad \text{there exists } g \in G \text{ such that } g(x_1) = x_2.$$

We have the quotient map

$$\pi : X \to X/G \quad \text{given by} \quad x \mapsto G \cdot x = G\text{-orbit of } x.$$

If

$$X \text{ is a topological space} \quad \text{and} \quad G \leq \text{Homeo}(X),$$

then X/G is given with the quotient topology.

Definition 4.9 (fixed-point free and properly discontinuous G-actions)**.** Let X be a topological space and $G \leq \text{Homeo}(X)$.

- **Fixed-point free:** The G-action on X is fixed-point free if and only if

$$\text{for any } x \in X, g \in G \setminus \{e\}, \quad \text{we have} \quad g(x) \neq x,$$

  i.e. no element of G other than the identity e has a fixed point

- **Properly discontinuous:** The G-action on X is properly discontinuous if and only if

$$\text{for any } x \in X, \quad \text{there exists a neighbourhood } U \text{ of } x \in X \quad \text{such that}$$
$$\text{for any } g \in G \setminus \{e\}, \text{ we have } g(U) \cap U = \emptyset,$$

i.e. $g_0(U) \cap g_1(U) = \emptyset$ whenever $g_0 \neq g_1$

Remark 4.4. If a G-action on X is properly discontinuous, then it is fixed-point free. The converse holds when G is finite and X is Hausdorff.

The converse above appears in Exercise 81.4 of Munkres. To be discussed.

Example 4.14. $\mathbb{Z}$ acting on $\mathbb{R}$ by translation is properly discontinuous.

Recall that $\mathbb{Z} \leq \mathbb{R}$ with the group operation being addition. Here, the group $G = \mathbb{Z}$ acts on $X = \mathbb{R}$ by translation. Specifically, for any $n \in \mathbb{Z}$, the action is given by

$$n \cdot x = n + x \quad \text{for } x \in \mathbb{R}.$$

This means that n shifts $x \in \mathbb{R}$ by n units. So, we need to find a neighbourhood U of $x \in \mathbb{R}$ such that for any $n \in \mathbb{Z}$, where $n \neq 0$, the translated neighborhood does not intersect with U, i.e. $(U + n) \cap U = \emptyset$. Take U to be a sufficiently small ε-neighbourhood around x, where $\varepsilon < 1/2$. So,

$$U = (x - \varepsilon, x + \varepsilon) \quad \text{where } 0 < \varepsilon < 1/2.$$

Then,

$$U + n = (x + n - \varepsilon, x + n + \varepsilon).$$

Clearly,

$$(U + n) \cap U = (x + n - \varepsilon, x + n + \varepsilon) \cap (x - \varepsilon, x + \varepsilon) = \emptyset \quad \text{for } \varepsilon < 1/2.$$

The choice of $\varepsilon < 1/2$ is obvious because

$$x + n - \varepsilon > x + \varepsilon \text{ if } n \geq 1 \quad \text{and} \quad x + n + \varepsilon < x - \varepsilon \text{ if } n \leq -1.$$

As we have constructed such a neighbourhood U, it follows that the $\mathbb{Z}$-action on $\mathbb{R}$ is properly iscontinuous.

Example 4.15. $\mathbb{Q}$ acting on $\mathbb{R}$ by translation is fixed-point free but not properly discontinuous.

Again, we recall that $\mathbb{Q} \leq \mathbb{R}$ with the group operation being addition. Here, the group $\mathbb{Q}$ acts on $\mathbb{R}$ by translation. Specifically, for any $q \in \mathbb{Q}$, the $\mathbb{Q}$-action is given by

$$q \cdot x = q + x \quad \text{for } x \in \mathbb{R}.$$

Similarly, this means that q shifts $x \in \mathbb{R}$ by q units. We first prove that the $\mathbb{Q}$-action on $\mathbb{R}$ is not properly discontinuous. Let U be an ε-neighbourhood around x, where $\varepsilon > 0$. So,

$$U = (x-\varepsilon, x+\varepsilon) \quad \text{and} \quad U+q = (x+q-\varepsilon, x+q-\varepsilon).$$

Let $q > 0$ be arbitrary (the case where $q < 0$ can be argued similarly based on what we did towards the end of our analysis of Example 4.14). Then, there exists $\varepsilon > q/2 > 0$ such that $(U+q) \cap U \neq \emptyset$. In fact, this argument is analogous to saying that there are infinitely many $q \in \mathbb{Q}$ satisfying the inequality $q < 2\varepsilon$ as $\mathbb{Q}$ is dense in $\mathbb{R}$.

Having said that, the $\mathbb{Q}$-action on $\mathbb{R}$ by translation, as mentioned, is fixed-point free. To see why, note that $q + x = x$, then we must have $q = 0$, i.e. the only element of $\mathbb{Q}$ that fixes any $x \in \mathbb{R}$ is $q = 0$, but $0 \in \mathbb{Q}$ is the identity element of $\mathbb{Q}$. It implies that for any non-identity element of $\mathbb{Q}$, we have $q \cdot x \neq x$.

Lemma 4.5. Let $p : E \to B$ be a covering map. Then, the action of $\text{Aut}(p) \subseteq \text{Homeo}(E)$ on E is properly discontinuous.

Proof. Since $p : E \to B$ is a covering map, then there exists a neighbourhood V of b_0 in B that is evenly covered by p. Let $U \subseteq p^{-1}(V)$ be the unique slice such that $e_0 \in U$. The action of $\text{Aut}(p)$ on each fiber is fixed-point free. Hence, for any $x \in U$ and $g \in \text{Aut}(p) \setminus \{e\}$, one has $g(x) \neq x$, so $g(x)$ lies in a slice of $p^{-1}(V)$ different from U. Hence. $g(U) \cap U = \emptyset$. □

Theorem 4.7. Let X be path-connected and locally path-connected, with $G \subseteq \text{Homeo}(X)$. Then,

the quotient map $\pi : X \to X/G$ is a covering map if and only if
the G-action on X is properly discontinuous.

In this case, $\pi : X \to X/G$ is a Galois covering map and $\text{Aut}(\pi) = G$ as subgroups of $\text{Homeo}(X)$.

Proof. The proof of the forward direction follows from Lemma 4.5.

As for the reverse direction. Suppose the G-action on X is properly discontinuous. For any open $U \subseteq X$,

$$\pi^{-1}(\pi(U)) = \bigcup_{g \in G} g(U) \text{ open in } X \quad \text{so} \quad \pi(U) \text{ is open in } X/G.$$

Given $G \cdot x \in X/G$, represented by $x \in X$, there exists an open neighbourhood U of x in X such that for any $g_0 \neq g_1$ in G, we have $g_0(U) \cap g_1(U) = \emptyset$. Then,

$$\pi^{-1}(\pi(U)) = \bigsqcup_{g \in G} g(U) \quad \text{is a disjoint union of the } g(U)\text{'s}.$$

For each $g \in G$, the map

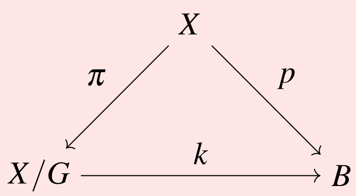
$$\pi : g(U) \to \pi(U) \quad \text{is bijective, continuous, open and hence a homeomorphism.}$$

Hence, $\pi : X \to X/G$ is a covering map. We have proven the reverse direction.

For any $x \in X$ and $g \in G$, one has $G \cdot g(x) = G \cdot x$ in X/G since x and $g(x)$ are in the same G-orbit. Hence, $\pi \circ g = \pi$, which implies $G \subseteq \text{Aut}(X)$. For any $h \in \text{Aut}(\pi)$, one has $\pi \circ h = \pi$ so $G \cdot h(x) = G \cdot x$ in X/G. As such, there exists $g \in G$ such that $g(x) = h(x)$ in X.

The action of $\text{Aut}(\pi)$ on each fiber is fixed-point free so $h = g$ lies in G. Hence, $\text{Aut}(\pi) = G$. Since $\text{Aut}(\pi) = G$ acts simply transitively on $\pi^{-1}(\pi(x))$, then $\pi : X \to X/G$ is a Galois covering map. □

Theorem 4.8. Let $p : X \to B$ be a Galois covering map. Let $G = \text{Aut}(p)$. Then, there exists a unique continuous map $k : X/G \to B$ such that the following diagram commutes:



Here, $\pi : X \to X/G$ is a quotient map. Moreover, k is a homeomorphism.

Proof. For any $g \in G$, one has $p \circ g = p$, so p is constant on each G-orbit. By the universal property of X/G, there exists a unique continuous map $k : X/G \to B$ such that $k \circ \pi = p$.

Note that k is surjective since p is surjective. k is injective since p is Galois (so $G = \text{Aut}(p)$ acts simply transitively on each fiber of p). Also, k is an open map as X/G is given with the quotient topology from $\pi : X \to X/G$ and p is an open map. Hence, k is a homeomorphism. □

Definition 4.10 (domination of covering space)**.** Let $p_0 : X \to B$ be a Galois covering map and $G = \text{Aut}(p_0)$. A covering space of B is dominated by X consists of

$$\text{a covering space } p : E \to B \quad \text{and} \quad \text{a } B\text{-space map } \pi : X \to E$$

Let X and X' be covering spaces of B dominated by X. A morphism from E to E' is a B-space map $k : E \to E'$ such that the following diagram commutes, i.e.

$$k \circ \pi = \pi' \quad \text{and} \quad p' \circ k = p$$

$$\begin{array}{ccc} X & & X \\ \pi \downarrow & & \downarrow \pi' \\ E & \longrightarrow & E' \\ p \downarrow & & \downarrow p' \\ B & & B \end{array}$$

Theorem 4.9. Let $p_0 : X \to B$ be a Galois covering map. Let $G = \operatorname{Aut}(p_0)$. Then, the map from the set of isomorphism classes of path-connected covering spaces of B dominated by X to the set of subgroups of G, where

$$\left(X \xrightarrow{\pi} E \xrightarrow{p} B\right) \mapsto \operatorname{Aut}(\pi)$$

is a well-defined order-reversing bijection.

Compare this with the following result:

Theorem 4.10. Let k be a field and let $k \hookrightarrow M$ be a Galois extension of k. Then, the map from the set of k-isomorphism classes of field extensions of k contained in M to the set of closed subgroups of $\operatorname{Gal}(M/k)$, where

$$M \mapsto \operatorname{Aut}(M/L),$$

is a well-defined order-reversing bijection. Here, $k \subseteq L \subseteq M$ are field extensions.

5. Fundamental Groups Revisited

5.1. *Fundamental Group of a Union*

Example 5.1. Consider the theta space

$$X = A \cup B \cup C \text{ which is the union of three arcs } \approx [0,1] \quad \text{with endpoints } p,q.$$

Then, X is homeomorphic to a graph.

Choose $a \in \text{int}(A), b \in \text{int}(B), c \in \text{int}(C)$. Also, let $U = X \setminus \{a\}$ and $V = X \setminus \{b\}$ so $X = U \cup V$. U has $B \cup C \approx \mathbb{S}^1$ as a deformation retract and V has $A \cup C$ as a deformation retract. So,

$$\pi_1(U, x_0) \cong \pi_1(V, x_0) \cong \mathbb{Z} \quad \text{which is an infinite cyclic group.}$$

So, $U \cap V = X \setminus \{a,b\}$ has C as a deformation retract, making it simply connected. Hence, $\pi_1(X, x_0) \cong \mathbb{Z} * \mathbb{Z}$.

there's something missing on free group missing stuff from previous lecture

Let $B^2 \subseteq \mathbb{R}^2$ be the unit ball and $\mathbb{S}^1 = \partial B^2$ (i.e. boundary of the ball). Suppose $\mathbf{p} \in \mathbb{S}^1$ and $\mathbf{0} \in \text{Int}\, B^2$ be the origin.

Theorem 5.1 (adjoining a two-cell)**.** Let X be a T_2 space and $A \subseteq X$ be a closed path-connected subspace. Let $h : B^2 \to X$ be a continuous map such that

$$h \text{ maps } \text{Int}\, B^2 \text{ bijectively onto } X \to A \quad \text{and} \quad h \text{ maps } \mathbb{S}^1 \text{ into } A.$$

So, $X = A \cup h\left(\text{Int}\, B^2\right)$.

Let $a = h(p) \in A$. Then,

$$\text{the homomorphism induced by inclusion} \quad i_* : \pi_1(A,a) \to \pi_1(X,a) \quad \text{is surjective.}$$

We have $\ker i_* \trianglelefteq \pi_1(A,a)$, generated by

$$\text{the image of} \quad h_* : \pi_1\left(\mathbb{S}^1, p\right) \to \pi_1(A,a).$$

Proof. Consider the open sets

$$V = X \setminus A = h\left(\text{Int}\, B^2\right) \quad \text{and} \quad U = X \setminus x_0.$$

Here, $x_0 = h(0) \in V$. The sets U and V are open since X is Hausdorff. Also,

$$U \cap V = V \setminus \{x_0\} = h\left(\text{Int}\, B^2 \setminus \{\mathbf{0}\}\right) \quad \text{is path-connected.}$$

The map

$$h \times \mathrm{id} : B^2 \times I \to h\left(B^2\right) \times I \quad \text{is closed}$$

since this is a map from a compact space to a subset of a Hausdorff space (since $h\left(B^2\right) \times I \subseteq X \times I$). So,

$$\begin{aligned} \text{the topology of } h\left(B^2\right) \times I \quad &\text{is the quotient topology from} \quad B^2 \times I \quad \text{and} \\ \text{the topology of } \left(h\left(B^2\right) \setminus \{x_0\}\right) \times I \quad &\text{is the quotient topology from} \ \left(B^2 \setminus \{\mathbf{0}\}\right) \times I \end{aligned}$$

The deformation retraction of the punctured closed disc $B^2 \setminus \{\mathbf{0}\}$ onto the unit circle $\mathbb{S}^1$ then induces a deformation retraction of $h\left(B^2\right) \setminus \{x_0\}$ onto $h\left(\mathbb{S}^1\right) \subseteq A$. We can extend this to a deformation retraction of $X \setminus \{x_0\}$ onto A by keeping each point of A fixed. So, A is a deformation retract of $U = X \setminus \{x_0\}$. It follows that

$$i_* : \pi_1(A, a) \to \pi_1(U, a) \quad \text{is an isomorphism induced by inclusion.}$$

Choose any $b \in U \cap V$ and $q = h^{-1}(b) \in \mathrm{Int}\left(B^2 \setminus \{\mathbf{0}\}\right)$. Let γ be any path in $B^2 \setminus \{\mathbf{0}\}$ from q to $p \in \mathbb{S}^1$. Then, $\delta = h \circ \gamma$ is a path in $U \cap V$ from b to $a \in A$.

Let f be a loop of $\mathbb{S}^1$ such that $[f]$ generates $\pi_1\left(\mathbb{S}^1, p\right)$. Then, $g = h \circ f$ is a loop in U based at a and $\delta * g * \widetilde{\delta} = h \circ \left(\gamma * f * \widetilde{f}\right)$ is a loop in U based at b. Hence,

$$\widetilde{\delta}^{-1}[g] = \left[\delta * g * \widetilde{\delta}\right] \in \pi_1(U_b) \quad \text{is the image of a generator of } \pi_1(U \cap V, b).$$

The result follows by the Seifert-van Kampen theorem. □

Theorem 5.2. The fundamental group of the torus has the presentation

$$\left\langle \alpha, \beta \mid \alpha\beta\alpha^{-1}\beta^{-1} \right\rangle \quad \text{which is a free Abelian group of rank 2.}$$

Definition 5.1 (dunce cap)**.** Let n be a positive integer with $n > 1$. Let

$$\mu_n = \left\{ e^{2k\pi i/n} : k \in \mathbb{Z} \right\} \subseteq \mathbb{S}^1 \quad \text{be the finite subgroup of the } n^{\text{th}} \text{ roots of unity.}$$

This group acts on $\mathbb{C}$ by multiplication. The n-fold dunce cap is the quotient space

$$X_n = B^2 / \mu_n \quad \text{of the unit ball } B^2 \text{ modulo the action of } \mu_n \text{ by rotation}$$

given with the quotient topology.

For any positive integer n with $n > 1$, let $r : \mathbb{S}^1 \to \mathbb{S}^1$ be rotation through the angle $2\pi/n$, i.e.

$$(\cos\theta, \sin\theta) \mapsto \left(\cos\left(\theta + \frac{2\pi}{n}\right), \sin\left(\theta + \frac{2\pi}{n}\right) \right).$$

Form a quotient space X from the unit ball by identifying each point $\mathbf{x}$ of $\mathbb{S}^1$ with the points $r(\mathbf{x}), r^2(\mathbf{x}), \ldots, r^{n-1}(\mathbf{x})$. Then, the compact Hausdorff space X is known as the n-fold dunce cap.

Theorem 5.3. The fundamental group of the n-fold dunce cap is cyclic of order n, i.e. it has the presentation

$$\langle \alpha \mid \alpha^n \rangle .$$

6. Classification of Surfaces

6.1. *Fundamental Groups of Surfaces*

Definition 6.1 (surface)**.** A surface is a connected topological 2-manifold.

We shall give a list of compact surfaces such that

(i) we can generate a good list of examples;

(ii) no two surfaces on the list are homeomorphic;

(iii) every compact surface is homeomorphic to one of them

So, we shall construct examples of compact connected surfaces as the quotient space obtained from a *polygonal region in the plane* by *pasting its edges together*.

Definition 6.2 (regular n-gon)**.** For any $n \geq 3$, the regular n-gon P_n is the convex hull in $\mathbb{R}^2$ of the points

$$p_k = \left(\cos \frac{2k\pi}{n}, \sin \frac{2k\pi}{n}\right) \quad \text{where } 0 \leq k \leq n \text{ and } p_n = p_0.$$

Definition 6.3 (polygonal region)**.** An n-sided polygonal region in the plane is any subset P of $\mathbb{R}^2$ which is given with a homomorphism from a regular n-gon P_n.

(i) The vertices/edges of P are the images of those of P_n;

(ii) The orientation of an edge e of P is an ordering ∂e from the initial point to the final point;

(iii) $\operatorname{Bd} P$ refers to the union of the edges of P, or equivalently, the images of the boundary of P_n;

(iv) $\operatorname{Int} P = P \setminus \operatorname{Bd} P$, which is the images of the interior of P_n

Up to homeomorphism, we may assume that $P = P_n$ without loss of generality.

Definition 6.4 (labels)**.** Let $P = P_n$ be an n-sided polygonal region in the plane. Let S be a set, where elements of S are known as labels.

(i) A labelling of the edges of P is a map from the set of edges of P to S

(ii) Given an orientation of each edge of P and a labelling of the edges of P, the corresponding labelling scheme w of length n is the sequence of labels with exponents $+1$ or -1

$$w = (a_{i_1})^{\varepsilon_1} (a_{i_2})^{\varepsilon_2} \dots (a_{i_n})^{\varepsilon_n}$$

where for each $1 \leq k \leq n$,

$$a_{i_k} = \text{label assigned to the edge } p_{k-1}p_k$$
$$\varepsilon_k = +1 \text{ or } -1$$

We note that $\varepsilon \in \{\pm 1\}$ according as the orientation assigned to this edge goes from p_{k-1} to p_k or the reverse.

Define the equivalence relation $\sim_w$ on P as follows:

$$x \sim_w y \quad \text{if and only if} \quad x = y.$$

This is in fact equivalent to saying that there exist edges e_x, e_y of P with the same label such that $x \in e_x, y \in e_y$ and $y = h(x)$, where h is the unique orientation-preserving linear map from e_x to e_y.

The space obtained by pasting the edges of P together according to the given orientations and labelling the given labelling scheme w is the quotient set $X = P/\sim_w$ of P modulo the equivalence relation $\sim_w$ given with the quotient topology.

Remark 6.1. A cyclic permutation of the terms in a labelling scheme will change the space X only up to homeomorphism.

Example 6.1 (constructing a circle from a triangle)**.** In Figure 16, the triangular shape has vertices labelled with colors (red, blue, green) and edges labelled a and b. The edges are oriented (with arrows) and identified as follows: two sides labelled a and one side labelled b. The path around the triangle corresponds to the word $a \cdot a^{-1} \cdot b$, which, when simplified, effectively identifies certain edges and points.

The triangle is folded according to the identification of edges. This quotient map (collapsing identified points) forms a cone-like structure, where the vertices meet at a single point (indicated by the arrow on top). The cone is shown to be homeomorphic to a closed disc. This means that by identifying edges in this way, we get a shape that can be continuously deformed into a disk.

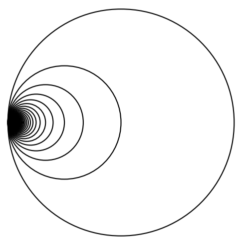


Figure 15: Hawaiian earring

Example 6.2 (constructing a sphere from a square)**.** In Figure 17, the square has four edges labelled with letters a and b, with pairs of opposite edges identified. Specifically, opposite a-edges and opposite b-edges are identified. The path around the boundary corresponds to $a^{-1}bb^{-1}a$ or equivalently $aa^{-1}bb^{-1}$, suggesting that opposite sides are glued together.

The square is transformed by identifying opposite edges, leading to a shape resembling a *lens* with two points (top and bottom) coming together. The resulting shape from identifying opposite sides in this way is homeomorphic to a sphere. This shows that a square with this particular edge identification can be transformed into a sphere through continuous deformation.

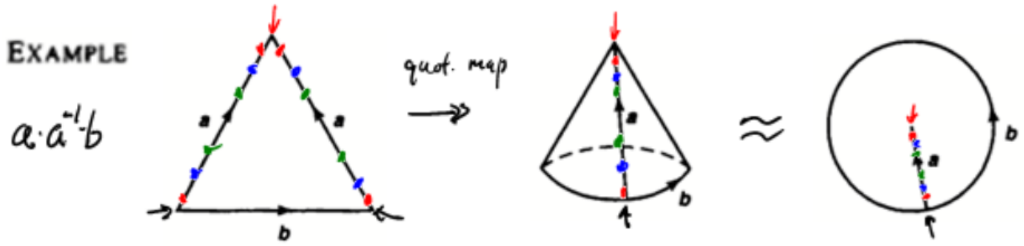


Figure 16: Constructing a circle from a triangle

Example 6.3 (constructing a torus and the dunce cap). We first discuss the top example of Figure 18, where we construct a torus from a square. This square has edges labelled a and b, with arrows indicating specific identifications.

By first identifying the two edges labelled a, we obtain a cylindrical shape. The two remaining edges labelled b are on the ends of the cylinder. Identifying the two b-edges results in a torus, a surface with a single hole. The torus is obtained by gluing the opposite edges of the square as per the labeling, which wraps the shape around to form a continuous loop with no boundary.

The bottom diagram of Figure 18 shows that we can construct the dunce cap from a square. This square also has edges labelled, but the path around the boundary is $abab$. Here, adjacent edges are identified in a way that does not preserve orientation.

When identifying the edges in this specific manner, the shape that emerges is the dunce cap, which is a type of quotient space with interesting topological properties. The Dunce cap has a relation that can be thought of as $P^2 = \mathbb{S}^2/\{\pm 1\}$, indicating a certain symmetry, but it does not embed into $\mathbb{R}^3$ smoothly.

The fundamental group of this space, $\pi_1\left(P^2, x_0\right) = \langle c \,|\, c^2 \rangle$, which is not isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$. This implies that the dunce cap has different homotopy properties compared to spaces like the torus, whose fundamental group is $\mathbb{Z} \oplus \mathbb{Z}$.

Given a finite number $P_1, \ldots, P_k$ of disjoint polygonal regions, along with orientation and labelling of their edges, one can form a quotient space X by pasting the edges of the regions together in exactly the same way.

Example 6.4 (constructing a Möbius band and a cylinder). Consider the top diagram in Figure 19. It shows a rectangle with four edges labelled a, b, c, and d. The edges are oriented, and we have expressions that describe how they should be identified. The relation $dbc^{-1}a^{-1}$ indicates specific edge pairings. Another relation, $fa^{-1}e^{-1}b^{-1}$, suggests additional identifications.

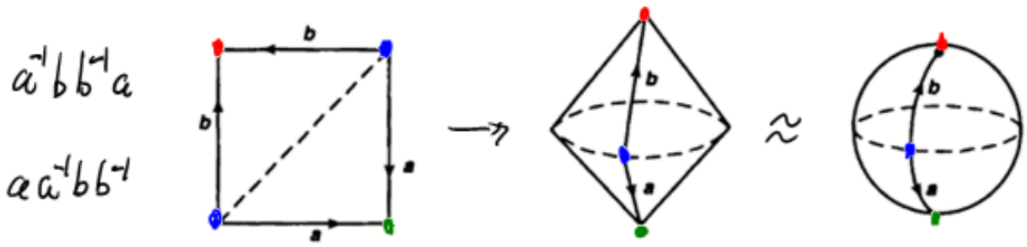


Figure 17: Constructing a sphere from a square

The rectangle's edges are manipulated to identify certain pairs. Specifically, edge a on one side is identified with edge a on the opposite side, but with a twist (indicated by the reversal of the arrow direction). By joining the edges a and b with a half-twist, we obtain a Möbius band. This is a non-orientable surface, meaning it has only one side. The twisting of the edges creates a unique property: if you follow a path along the surface, you'll eventually return to your starting point but on the other side of the band.

For the bottom diagram, the rectangle again has labelled edges (a, b, c, and d) with orientations. The relations here are $ac^{-1}a^{-1}d$ and $be^{-1}b^{-1}f$, describing how the edges should be glued. This time, edge a is identified with itself without any twists, resulting in a straightforward gluing. Similarly, edge b is identified with itself in a way that does not introduce any twists.

Identifying the edges in this way produces a cylinder, a surface with two boundaries (top and bottom) and an orientable structure. Unlike the Möbius band, the cylinder does not involve a twist, so it has two distinct sides.

To summarise, by identifying opposite edges with a twist, we obtain a non-orientable surface, the Möbius band, which has only one side; by identifying edges without a twist, we get a cylinder, an orientable surface with two distinct sides and boundaries.

Theorem 6.1. Let $P = P_n$ be an n-sided polygonal region in the plane. Le $X = P/\sim_w$ be the space obtained by pasting the edges of P together according to a labelling scheme w.

Theorem 6.2. Suppose π maps all the vertices of P to a single point $x_0 \in X$. Let $a_1, \ldots, a_k$ be the distinct labels (k many) in the labelling scheme w. Then, $\pi_1(X, x_0)$ has the presentation

$$\left\langle a_1 \ldots a_k \mid (a_{i_1})^{\varepsilon_1} \ldots (a_{i_n})^{\varepsilon_n} \right\rangle.$$

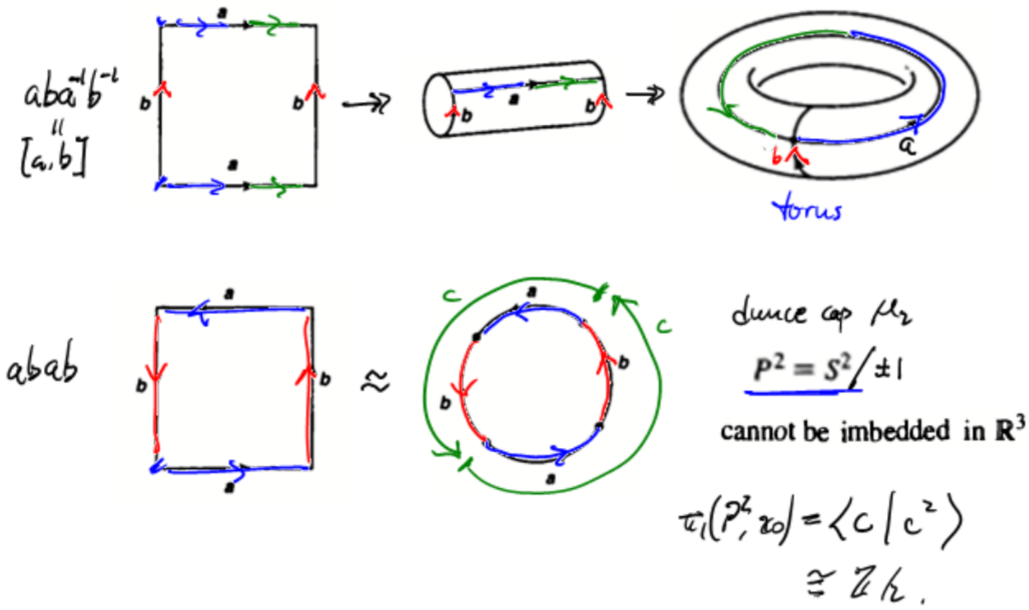


Figure 18: Constructing a torus and the dunce cap

Definition 6.5 (n-fold torus)**.** For any $n \geq 1$, the n-fold torus is the space obtained by pasting the edges of a $4n$-sided regular polygonal region P according to the labelling scheme

$$\left(a_1 b_1 a_1^{-1} b_1^{-1}\right)\left(a_2 b_2 a_2^{-1} b_2^{-1}\right)\ldots\left(a_n b_n a_n^{-1} b_n^{-1}\right) \quad \text{where } [a_i, b_i] = a_i b_i a_i^{-1} b_i^{-1} \text{ is the commutator of } a_i \text{ and } b_i.$$

This is also called the connected sum of n tori, denoted by $\mathbb{T}\#\ldots\#\mathbb{T}$.

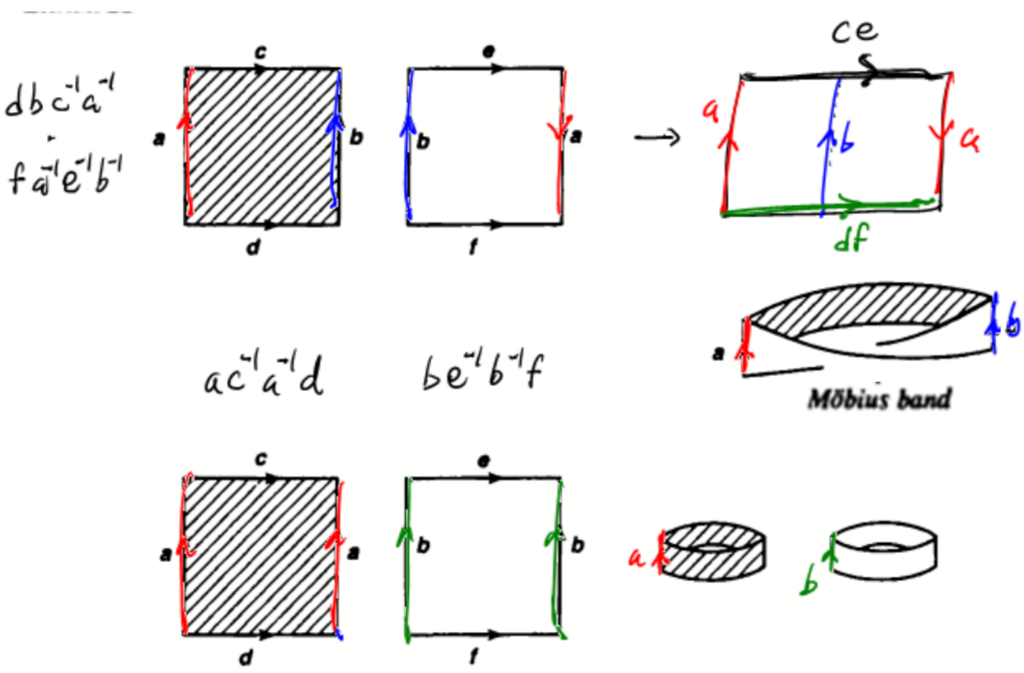


Figure 19: Constructing a Möbius band and a cylinder

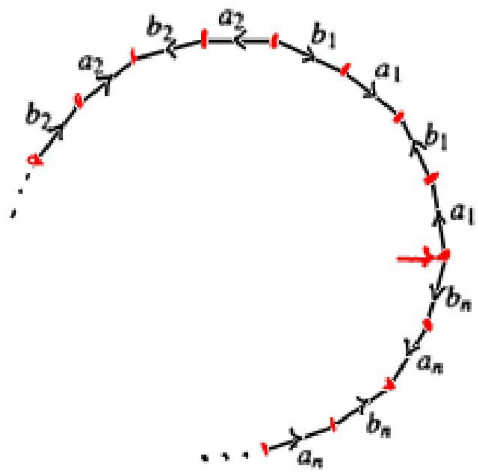


Figure 20: Labelling of polygon edges

Example 6.5 (2-fold torus)**.** On the left diagram of Figure 21[†], we have a polygonal schema that is often used to represent surfaces with higher genus (genus refers to number of holes). The edges labeled a_1, b_1, a_2, b_2 correspond to loops that represent generators of the fundamental group. Each pair of edges with the same label are meant to be identified or *glued together* in a specific orientation, effectively forming the shape of a 2-fold torus.

The central curve c represents a *cut*. This representation helps derive the fundamental group, where the relations among a_1, b_1, a_2, b_2 combine to give the fundamental group of the 2-fold torus, which is

$$\langle a_1, b_1, a_2, b_2 \mid [a_1, b_1][a_2, b_2] = 1 \rangle .$$

The two diagrams in the middle depict a single torus with two loops, marked in red and blue. The rightmost diagram shows a connected sum notation $\mathbb{T}\#\mathbb{T}$, indicating the connected sum of two tori to create a 2-fold torus. Here, a connected sum operation joins two surfaces by removing a disk from each and gluing them together along the resulting boundary circles.

The diagram of the 2-fold torus shows two holes, corresponding to the two tori that have been joined. The blue loop marked on the 2-fold torus indicates one of the fundamental loops, a part of the set of generators that describe the fundamental group for this surface.

Example 6.6 (3-fold torus)**.** Similarly, we can obtain the 3-fold torus from a regular dodecagon (12-sided polygon) as shown in Figure 22.

Theorem 6.3 (fundamental group of n-fold torus)**.** Let X denote the n-fold torus. Then, $\pi_1(X, x_0)$ has the presentation

$$\langle \alpha_1, \beta_1, \ldots, \alpha_n, \beta_n \mid [\alpha_1, \beta_1] \ldots [\alpha_n, \beta_n] \rangle .$$

[†]Here is a YouTube video which shows how one can visualise the construction of a 2-fold torus from an octagon

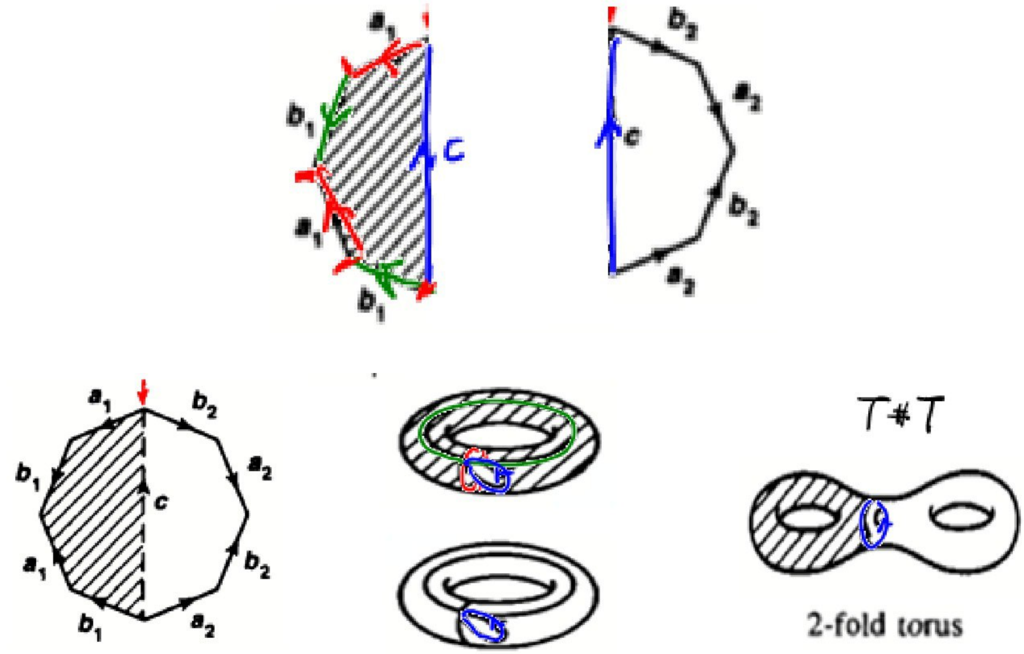


Figure 21: The 2-fold torus $\mathbb{T}\#\mathbb{T}$

Here, the presentation has $2n$ generators and a single relation.

Definition 6.6 (m-fold projective plane)**.** For any $m > 1$, the m-fold projective plane is the space obtained by pasting the edges of a $2m$-sided polygonal region P according to the labelling scheme

$$(a_1 a_1)(a_2 a_2)\dots(a_m a_m).$$

This is also called the connected sum of p projective planes, denoted by $\mathbb{P}^2\#\dots\#\mathbb{P}^2$.

Example 6.7 (2-fold projective plane)**.** Figure 23 indicates how the connected sum of two projective planes, denoted by $\mathbb{P}^2\#\mathbb{P}^2$, can be obtained from two copies of the projective plane by deleting an open disc from each and pasting the resulting spaces together along the boundaries of the deleted discs.

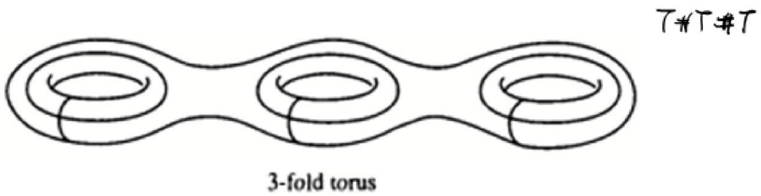


Figure 22: The 3-fold torus $\mathbb{T}\#\mathbb{T}\#\mathbb{T}$

As with $\mathbb{P}^2$ itself, we have no convenient way for picturing the m-fold projective plane as a surface in $\mathbb{R}^3$, for in fact it cannot be imbedded in $\mathbb{R}^3$ (see Example 6.8 on the Klein bottle).

Example 6.8 (Klein bottle)**.** Figure 24 illustrates the construction of a Klein bottle through edge identifications. On the left diagram, the square represents the polygonal schema for a surface where opposite edges are identified to create a Klein bottle. The edges are labeled a and b with specific orientations — the top and bottom edges are labelled a and oriented in the same direction; the left and right edges are labeled b but oriented in opposite directions.

The relation $aba^{-1}b$ shown beneath the square is a presentation for the fundamental group of the Klein bottle.

For the diagram in the centre, the edges labelled a are glued together to form a cylinder. The green arrows on the circular boundaries of the cylinder represent the b edges, which are yet to be identified. To complete the construction of the Klein bottle, these b edges need to be glued together, but with a twist, as indicated by the opposing directions of the green arrows. This twist is what gives the Klein bottle its non-orientable property.

The rightmost diagram shows the completed Klein bottle after the twisted identification of the b edges. The path of the a edge has become a continuous loop within the Klein bottle, and the twisted b identification has introduced a self-intersection, which is necessary when visualising the Klein bottle in $\mathbb{R}^3$ †.

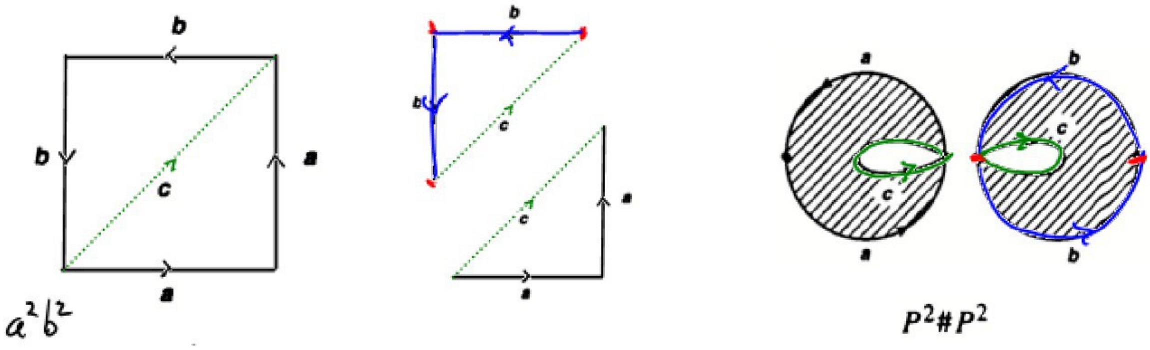


Figure 23: The connected sum of two projective planes $\mathbb{P}^2\#\mathbb{P}^2$

Figure 25 shows a two-dimensional representation of the Klein bottle embedded in $\mathbb{R}^3$.
In Group theory, the Baumslag-Solitar groups are examples of two-generators one-relation groups.

Definition 6.7 (Baumslag-Solitar group)**.** The Baumslag-Solitar groups have the presentation

$$\mathrm{BS}(m,n) = \left\langle a,b \mid ba^m b^{-1} = a^n \right\rangle.$$

†Here is a YouTube video which shows how one can visualise the construction of the Klein bottle from a square

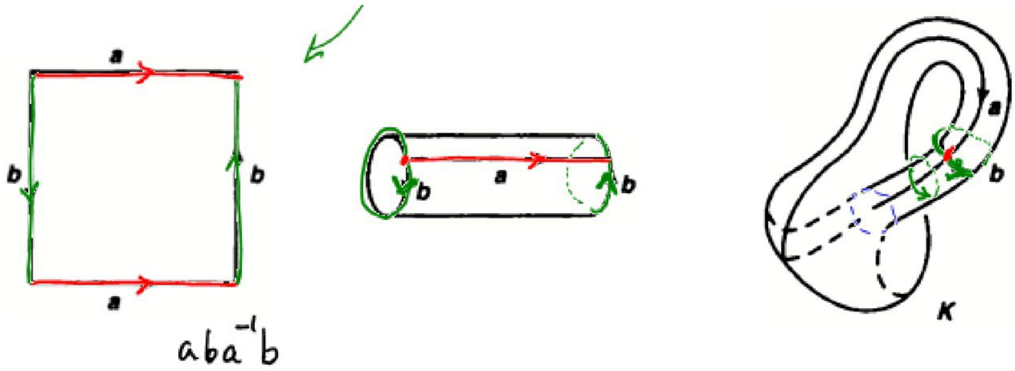


Figure 24: Obtaining the Klein bottle from a square

Example 6.9. BS $(1,1)$ is the free Abelian group on two generators; BS $(1,-1)$ is the fundamental group of the Klein bottle (see Example 6.10 where the Munkres textbook presents a question on this).

Example 6.10 (Munkres p. 454 Question 3)**.** The Klein bottle K is the space obtained from a square by means of the labelling scheme $aba^{-1}b$. Figure 24 indicates how K can be pictured as an immersed surface in $\mathbb{R}^3$.

(a) Find a presentation for the fundamental group of K.

(b) Find a double covering map $p : \mathbb{T} \to K$, where $\mathbb{T}$ is the torus. Describe the induced homomorphism of fundamental groups.

Solution.

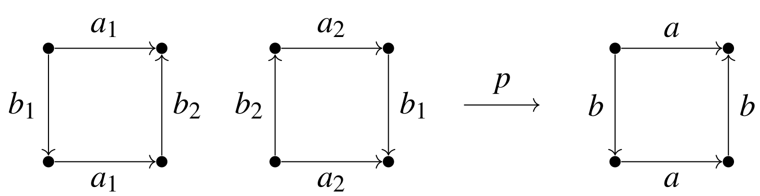
(a) As shown in the left diagram of Figure 24, in a presentation of the fundamental group of the Klein bottle, it should contain the generators are a and b with the relation $aba^{-1}b = 1$. As such,

$$\pi_1(K) = \langle a, b \mid aba^{-1}b \rangle .$$

(b) Each diagram on the left represents a torus $\mathbb{T}$, whereas the diagram on the right represents the Klein bottle K. So, we can define the covering map $p : \mathbb{T} \to K$ as follows:

$$p([s,t]) = \begin{cases} [2s, t] & \text{if } 0 \leq t \leq 1/2; \\ [2s-1, 1-t] & \text{if } 1/2 \leq t \leq 1. \end{cases}$$

So, the map p takes the first square directly to the square on the right, and it flips the second square vertically (along the middle horizontal line). By the pasting lemma, this map p is continuous on the image of each of the two squares and agrees on the overlap.



We then compute the induced homomorphism p_*, where

$$p_* : \pi_1(\mathbb{T}, x_0) \to \pi_1(K, y_0) .$$

Observe that

$$p_*(a_1) = p_*(a_2) = a \quad \text{and} \quad p_*(b_1) = p_*(b_2) = b.$$

Note that $\alpha = a_1 a_2$ and $\beta = b_1$ are loops. As $p_*(\alpha) = a^2$ and $p_*(\beta) = b$ and the relation $aba^{-1}b = 1$ in the presentation of K ensures that the two equations induce a homomorphism on $\pi_1(\mathbb{T}, x_0)$, i.e. $p_*(\alpha) \cdot p_*(\beta) = p_*(\beta) \cdot p_*(\alpha)$. □

Theorem 6.4 (fundamental group of m-fold projective plane)**.** Let X denote the m-fold projective plane. Then, $\pi_1(X, x_0)$ has the presentation

$$\left\langle \alpha_1, \ldots, \alpha_m \mid (\alpha_1)^2 \ldots (\alpha_m)^2 \right\rangle.$$

Here, the presentation is given by m generators and a single relation.

6.2. *Homology of Surfaces*

We wish to give a good classification, i.e. give a list of compact surfaces such that the surfaces obtained by pasting the edges of disjoint polygonal regions, according to labelling schemes $w_1, \ldots, w_m$, are such that no two surfaces on the list are homeomorphic and every compact surface is homeomorphic to one of them.

The spaces of interest are the 2-sphere $\mathbb{S}^2$, the n-fold torus $\mathbb{T}\#\ldots\#\mathbb{T}$ where $n \geq 1$, and the m-fold projective plane $\mathbb{P}^2\#\ldots\#\mathbb{P}^2$ where $m \geq 1$.

Recall from MA2202 that there is no effective procedure in general for the isomorphism problem for groups, i.e. given two finite presentations of groups, it is very difficult to determine whether or not the groups are isomorphic. However, for finitely generated groups with one relation, we can consider the abelianization of a group, which is defined as follows:

Definition 6.8 (abelianization)**.** The abelianization of a group G is the maximal Abelian quotient

$$G^{\text{ab}} = G/[G, G] \quad \text{of } G.$$

Definition 6.9 (abelianized fundamental group)**.** Let X be a path-connected topological space. The abelianized fundamental group of X is

$$\pi_1^{\text{ab}}(X) = \pi_1(X, x_0)^{\text{ab}} = \pi_1(X, x_0) / [\pi_1(X, x_0), \pi_1(X, x_0)]$$

for any choice of base point $x_0 \in X$.

Lemma 6.1. For any $x_1 \in X$, choose a path α in X from x_0 to x_1. The isomorphism

$$\widehat{\alpha} : \pi_1(X, x_0) \to \pi_1(X, x_1) \quad \text{where} \quad [f] \mapsto \alpha([f]) = [\widetilde{\alpha}] * [f] * [\alpha]$$

induces the same isomorphism

$$\pi_1(X, x_0)^{\text{ab}} \cong \pi_1(X, x_1)^{\text{ab}} \quad \text{independent of the choice of the path } \alpha.$$

Proof. Suppose β is another path in X from x_0 to x_1. Then, $g = \alpha * \widetilde{\beta}$ is a loop in X at x_0, so $[g] \in \pi_1(X, x_0)$. As such, $\widehat{g} : \pi_1(X, x_0) \to \pi_1(X, x_0)$ given by $[f] \mapsto \widehat{g}[f] = [g]^{-1} * [f] * [g]$ (conjugation by $[g]^{-1}$) induces the identity isomorphism on $\pi_1(X, x_0)^{\text{ab}}$. Hence, $\widehat{\alpha} = \widehat{\beta}$. □

The homology groups $H_n(X)$ of X for all $n \geq 0$ are defined in Homology Theory (covered in a graduate course in Algebraic Topology). Moreover, a graduate course would cover theorems such as Hurewicz theorem, which states that $H_1(X) \cong \pi_1^{\text{ab}}(X)$ canonically.

Theorem 6.5 (universal property of abelianization)**.** Let F be a group and $N \trianglelefteq F$. Let

$$p : F \to F^{\text{ab}} = F/[F, F] \quad \text{be the quotient homomorphism.}$$

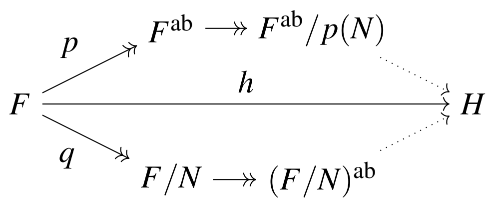
Then,

$$F^{\text{ab}}/p(N) \cong (F/N)^{\text{ab}} \quad \text{canonically.}$$

Proof. Both $F^{\text{ab}}/p(N)$ and $(F/N)^{\text{ab}}$ are Abelian groups with the same universal property. For any homomorphism $h : F \to H$ from any group F to an Abelian group H such that $N \subseteq \ker h$, there exists unique homomorphism

$$F^{\text{ab}}/p(N) \to H \quad \text{and} \quad (F/N)^{\text{ab}} \to H$$

such that the following diagram commutes:



The result follows. □

Corollary 6.1. Let G be a group with the presentation $\langle \alpha_1, \ldots, \alpha_n \mid x \rangle$, where $\alpha, \ldots, \alpha_n$ are the n generators and x is the relation. Let

$$F = \langle \alpha_1, \ldots, \alpha_n \rangle \quad \text{be the free group generating } G.$$

Define $p : F \to F^{\text{ab}} = \mathbb{Z}^n$ to be the quotient homomorphism. Then, $G^{\text{ab}} \cong \mathbb{Z}^n/p(x)$ canonically.

Proof. Write $G = F/N$, where $N = \langle x \rangle \trianglelefteq F$ is the normal subgroup of F generated by $\langle x \rangle$. The result follows by Theorem 6.5. □

Theorem 6.6 (fundamental group of n-fold torus)**.** Let X denote the n-fold torus $\mathbb{T}\#\ldots\#\mathbb{T}$. Then,

$$\pi_1^{\text{ab}}(X) \cong \mathbb{Z}^{2n} \quad \text{is free Abelian of rank } 2n.$$

Proof. Recall that $\pi_1(X,x)$ has the presentation

$$\langle \alpha_1, \beta_1, \ldots, \alpha_n, \beta_n \mid [\alpha_1, \beta_1] \ldots [\alpha_n, \beta_n] \rangle.$$

The element $[\alpha_1, \beta_1] \ldots [\alpha_n, \beta_n] \in \langle \alpha_1, \beta_1, \ldots, \alpha_n, \beta_n \rangle$ maps to 0 in $\mathbb{Z}^{2n}$. □

Theorem 6.7. Let X denote the m-fold projective plane $\mathbb{P}^2\#\ldots\#\mathbb{P}^2$. Then,

$$\pi_1^{\text{ab}}(X) \cong \mathbb{Z}^{m-1} \oplus \mathbb{Z}/2\mathbb{Z}$$

has torsion subgroup of order 2 and it is free quotient of rank $m-1$ (Copy phrasing from Munkres).

Proof. Recall that $\pi_1(X,x_0)$ has the presentation

$$\left\langle \alpha_1, \ldots, \alpha_m \mid (\alpha_1)^2 \ldots (\alpha_m)^2 \right\rangle.$$

The element $(\alpha_1)^2 \ldots (\alpha_m)^2$ maps to $p(x) = 2(\alpha_1 + \alpha_m) \in \mathbb{Z}^m$. Changing bases in $\mathbb{Z}^m$ to $\alpha_1, \ldots, \alpha_{m-1}$, with $\beta = \alpha_1 + \ldots + \alpha_m$, we have $p(x) = 2\beta$, so $\pi_1^{\text{ab}}(X) \cong (\mathbb{Z}\alpha_1 \oplus \ldots \oplus \mathbb{Z}\alpha_{m-1}) \oplus \mathbb{Z}\beta/2\beta$.

Here, we note that $\mathbb{Z}\alpha_1 \oplus \ldots \oplus \mathbb{Z}\alpha_{m-1} \cong \mathbb{Z}^{m-1}$ and $\mathbb{Z}\beta/2\beta \cong \mathbb{Z}/2\mathbb{Z}$. □

Theorem 6.8. Let $\mathbb{T}_n$ and $\mathbb{P}_m$ denote the n-fold connected sum of tori and the m-fold connected sum of projective planes, respectively. Then the surfaces $\mathbb{S}^2; \mathbb{T}_1, \mathbb{T}_2, \ldots; \mathbb{P}_1, \mathbb{P}_2, \ldots$ are topologically distinct.

6.3. *Cutting and Pasting*

As shown in Figure 26, let P be a polygonal region with successive vertices $p_0, \ldots, p_n$, where $p_n = p_0$. For $1 < k < n-1$, let

Q_1 = the polygonal region with successive vertices $p_0, p_1, \ldots, p_k, p_0$

Q_2 = the polygonal region with successive vertices $p_0, p_k, \ldots, p_n = p_0$

where they have the edge $p_0 p_k$ in common.

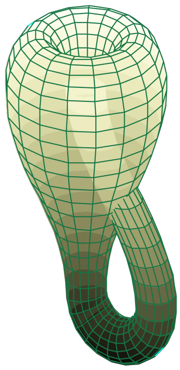


Figure 25: The Klein bottle

Define Q_1' to be any translation of Q_1 disjoint from Q_2 with successive vertices $q_0, q_1, \ldots, q_k, q_0$.

Then, P is homeomorphic to the quotient space of Q_1', Q_2 obtained by pasting the edge $\overrightarrow{q_0 q_k}$ of Q_2 to the edge $\overrightarrow{p_0 p_k}$ of Q_1'.

Say P has the labelling scheme $y_0 y_1$. Then, Q_1' has the labelling scheme $y_0 c^{-1}$ and Q_2 has the labelling scheme $c y_1$. Here, y_0 refers to the first k terms, y_1 is the remainder, and c is a label not in any labelling scheme elsewhere.

Elementary operations on labelling schemes of disjoint polygonal regions involve the following:

(i) Cut: One can replace the scheme $w_1 = y_0 c y_1$ by the scheme $y_0 c^{-1}$ and $c y_1$, provided c does not appear elsewhere in the total scheme and y_0 and y_1 have length at least two.

(ii) Paste: One can replace the scheme $y_0 c^{-1}$ and $c y_1$ by the scheme $y_0 y_1$, provided c does not appear elsewhere in the total scheme.

(iii) Relabel: One can replace all occurrences of any given label by some other label that does not appear elsewhere in the scheme. Similarly, one can change the sign of the exponent of all occurrences of a given label a; this amounts to reversing the orientations of all the edges labelled "a". Neither of these alterations affects the pasting map.

(iv) Permute: One can replace any one of the schemes w_i by a cyclic permutation of w_i. Specifically, if $w_i = y_0 y_1$, we can replace w_i by $y_1 y_0$. This amounts to renumbering the vertices of the polygonal region P_i so as to begin with a different vertex; it does not affect the resulting quotient space.

(v) Flip: One can replace the scheme

$$w_i = (a_{i_1})^{\varepsilon_1} \cdots (a_{i_n})^{\varepsilon_n}$$

by its formal inverse

$$w_i^{-1} = (a_{i_n})^{-\varepsilon_n} \cdots (a_{i_1})^{-\varepsilon_1}.$$

This amounts simply to "flipping the polygonal region P_i over." The order of the vertices is reversed, and so is the orientation of each edge. The quotient space X is not affected.

(vi) Cancel: One can replace the scheme $w_i = y_0 a a^{-1} y_1$ by the scheme $y_0 y_1$, provided a does not appear elsewhere in the total scheme and both y_0 and y_1 have length at least two (Figure 27).

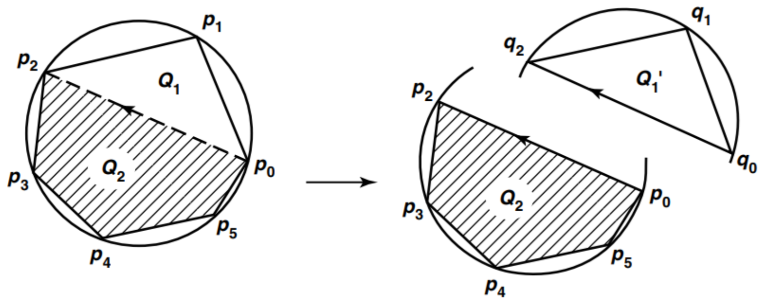


Figure 26

(vii) Uncancel: This is the reverse of operation **(vi)**. It replaces the scheme $y_0 y_1$ by the scheme $y_0 a a^{-1} y_1$, where a is a label that does not appear elsewhere in the total scheme. We shall not actually have occasion to use this operation.

Definition 6.10 (equivalent labelling schemes)**.** Two labelling schemes for disjoint polygonal regions are equivalent if and only if one can be obtained from the other by a finite sequence of elementary operations.

Note that the spaces obtained by pasting the edges of disjoint polygonal regions according to equivalent labelling schemes are homeomorphic to each other.

Example 6.11 (labelling scheme of Klein bottle). We note that the Klein bottle K is homeomorphic to the connected sum of the projective planes, i.e.

$$K \quad \text{is homeomorphic to} \quad \mathbb{P}^2 \# \mathbb{P}^2$$

because their labelling schemes are equivalent.

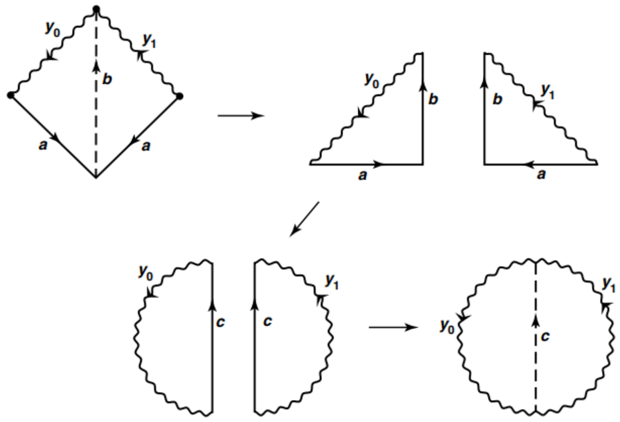


Figure 27

6.4. *The Classification Theorem*

Recall that the examples $\mathbb{S}^2, \mathbb{T}_n$ (where $n \geq 1$) and $\mathbb{P}_m$ (where $m \geq 1$) are obtained by pasting the edges in pairs from a single polygonal region. Our current goal is to provide a topological classification of surfaces obtained by quotients of a polygonal region.

Theorem 6.9 (classification theorem). Let X be the space obtained by pasting the edges in pairs from a single polygonal region. Then, X is homeomorphic to either of the following:

$$\text{the 2-sphere } \mathbb{S}^2 \quad \text{or} \quad \text{the } n\text{-fold torus } \mathbb{T}_n \quad \text{or} \quad \text{the } m\text{-fold projective plane } \mathbb{P}_m,$$

where $n, m \geq 1$.

Definition 6.11 (proper labelling scheme). A labelling scheme $w_1, \ldots, w_k$ of disjoint polygonal regions $P_1, \ldots, P_k$ is proper if and only if each label appears exactly twice in $w_1, \ldots, w_k$.

Theorem 6.10 (classification of proper labelling schemes). Let w be a proper labelling scheme of a single polygonal region of length ≥ 4. Then, w is equivalent to either of the following:

(1) $aa^{-1}bb^{-1}$

(2) $abab$

(3) $(a_1a_1)\ldots(a_ma_m)$ where $m \geq 2$

(4) $\left(a_1 b_1 a_1^{-1} b_1^{-1}\right) \ldots \left(a_n b_n a_n^{-1} b_n^{-1}\right)$ where $n \geq 1$

Definition 6.12. Let w be a proper labelling scheme of a single polygonal region. Then,

w is of torus type if and only if each label appears exactly once with exponent ± 1.

Otherwise, w is of projective type.

Lemma 6.2. If $w = [y_0]\, a\, [y_1]\, a\, [y_2]$, then $w \sim aa\left[y_0 y_1^{-1} y_2\right]$. Here, w denotes a proper labelling scheme of a single polygonal scheme, and y_i denotes labelling schemes that may be empty, whereas a denotes a label.

Corollary 6.2. If w is a proper labelling scheme of projective type, then w is equivalent to a scheme of the same length of the form

$$(a_1 a_1) \ldots (a_k a_k)\, w_1 \quad \text{where } k \geq 1 \text{ with } w_1 \text{ of torus type or empty.}$$

Corollary 6.3. If $w = w_0\,(cc)\left(aba^{-1}b^{-1}\right) w$, then $w \sim w_0\,(aabbcc)\, w_1$.

Lemma 6.3. If $w = w_0 w_1$, where w_1 is of torus type with no two adjacent terms having the same label, then $w \sim w_0 w_2$, where $w_2 = aba^{-1}b^{-1}w_3$ with w_2 having the same length as w_1 and w_3 of torus type or empty.

6.5. *Constructing Compact Surfaces*

It suffices to establish the following theorem:

Theorem 6.11. Let X be a compact connected surface. Then, X is

homeomorphic to the space obtained by pasting the edges in pairs from a single polygonal region.

Will use the following result:

Theorem 6.12 (Rado's theorem)**.** Let X be a compact surface. Then, X is homeomorphic to the space

obtained by pasting the edges in pairs from a finite collection of disjoint triangular regions.

We now prove Theorem 6.11.

Proof. By Rado's theorem (Theorem 6.12), there exists a finite collection of disjoint polygonal triangular regions $T_1, \ldots, T_n$, and a proper labelling scheme of these regions such that X is homeomorphic to the space obtained by pasting the edges of these regions according to the labelling scheme.

If there exist two edges in distinct regions with the same label, then by flipping one of the regions if necessary, we can paste these regions along these two edges. As such, the number of polygonal regions is reduced by 1, making the labelling scheme still proper.

By induction, this is reduced to the situation where either one has only a single polygonal region with proper labelling scheme (for which we are done) or one has more than one polygonal region such that no two distinct regions missing! □

Definition 6.13 (curved triangle)**.** Let T be the regular T-gon in $\mathbb{R}^2$. Let X be a topological space. A curved triangle in X is

$$\text{a subspace } A \subseteq X \quad \text{given with} \quad \text{a homeomorphism } h : T \to A.$$

The vertices and edges of A are the images of those of T.

Definition 6.14 (triangulation)**.** A triangulation of X is a collection $\{A_\alpha\}_\alpha$ of subspaces of X satisfying the following properties:

(i) for each α, the subspace A_α is a curved triangle in X, so homeomorphic to T

(ii) for each $\alpha \neq \beta$, we have

$$A_\alpha \cap A_\beta = \begin{cases} \emptyset; \\ \text{a vertex of both } A_\alpha, A_\beta; \\ \text{an edge of both } A_\alpha, A_\beta \end{cases}$$

and if $e = A_\alpha \cap A_\beta$ is an edge of both and

$$h_\alpha : T \to A_\alpha \quad \text{and} h_\beta : T \to A_\beta \quad \text{are the given homeomorphisms,}$$

then

$$h_\beta^{-1} h_\alpha : h_\alpha^{-1}(e) \to h_\beta^{-1}(e) \quad \text{is a linear homeomorphism between the edges of } T$$

(iii)

$$X = \bigcup_\alpha A_\alpha \quad \text{and it is given with the colimit topology.}$$

Definition 6.15 (triangularizable)**.** The space

$$X \text{ is triangularizable} \quad \text{if and only if} \quad \text{there exists a triangulation of } X.$$

We now prove Rado's theorem (Theorem 6.12).

Proof. Since X is compact and Hausdorff, then we can write

$$X = \bigcup_{i=1}^{n} A_i \quad \text{which is a finite union of curved triangles.}$$

For each i, let

$$h_i : T_i \to A_i \quad \text{denote} \quad \text{the corresponding homeomorphism.}$$

Here, T_i's are disjoint regular 3-gons in $\mathbb{R}^2$. Also, if $e = A_i \cap A_j$ is an edge of both, then

$$h_j^{-1} h_i : h_i^{-1}(e) \to h_j^{-1}(e) \quad \text{is a linear homeomorphism.}$$

Let

$$E = \bigcup_{i=1}^{n} T_i \subseteq \mathbb{R}^2 \quad \text{be a finite collection of disjoint triangular regions.}$$

Then, the h_i's combine to define a surjective map $h : E \to X$. Since E is compact and X is Hausdorff, then the map h is closed, so the topology of X is the quotient topology from E via h.

For each edge e of a curved triangle A_i, there exists a unique different curved triangle A_j with e as an edge. This follows from the fact that each point on the edge in question must have an open neighbourhood homeomorphic to the open disc in $\mathbb{R}^2$. Hence, $h : E \to X$ pastes the edges of E in pairs.

Let w be the labelling scheme on the edges of E induced by h. Then, w is a proper labelling scheme. Recall that the quotient space $E/\sim_w$ is obtained by pasting the edges of E together according to the labelling scheme w. It is defined via the equivalence relation $\sim_w$ (check) on E generated by the following relation:

$$\begin{aligned} x = y \quad \text{or} \quad & \text{there exist edges } e_x, e_y \text{ of } E \text{ with the same label} \\ & \text{such that } x \in e_x \text{ and } y \in e_y \text{ and } y = \phi(x) \\ & \text{where } \phi \text{ is the unique orientation-preserving linear map from } e_x \text{ to } e_y \end{aligned}$$

From the definition of triangulation of X, it is clear that for any $x, y \in E$, we have

$$x \sim_w y \quad \text{implies} \quad h(x) = h(y) \text{ in } X$$

and the reverse implication holds when x, y are in the interior of the T_i's or in the interior of the edges. However, it is not clear that

$$h(x) = h(y) \text{ in } X \text{ when } x, y \text{ are the vertices of } T_i\text{'s} \quad \text{implies} \quad x \sim_w y.$$

For any curved triangles A_i, A_j such that $v = A_i \cap A_j$ is a vertex of both, then there exists a sequence of curved triangles with v as vertex, beginning with A_i and ending with A_j such that

the intersection of each triangle of the sequence and its successor is an edge of both.

Then, X is indeed the space $E/\sim_w$ obtained by pasting the edges of E together according to the proper labelling scheme w induced by h. To see why, let v be a vertex of any curved triangle. Define the relation $\sim$ on the set of curved triangles with v as a vertex by setting

$A_i \cap A_j$ if and only if there exists a sequence of curved triangles
with v as vertex beginning with A_i and ending with A_j
such that missing
is an edge of both

It is clear that $\sim$ is an equivalence relation and we wish to show that there is only one equivalence class. Suppose otherwise. Let B be the union of the curved triangles in one equivalence class and C be the union of all the others. Then, $B \cup C$ contains a neighbourhood of v in X.

Also, no triangle in B has an edge in common with any triangle in C so

$$B \cap C = \{v\} \quad \text{so} \quad (B \cup C) \setminus v \text{ is not connected.}$$

Hence, for every sufficiently small neighbourhood W of v in X (i.e. $W \subseteq B \cup C$), the space $W \setminus v$ is not connected.

include diagram

However, since X is a surface, it is locally homeomorphic to $\mathbb{R}^2$, i.e. there exists a neighbourhood U of v in x that is homeomorphic to $\mathbb{R}^2$, but

$$U \setminus v \quad \text{is homeomorphic to} \quad \mathbb{R}^2 \setminus \{\mathbf{0}\},$$

where it is known that $\mathbb{R}^2 \setminus \{\mathbf{0}\}$ is connected, which is a contradiction. missing □

7. Applications of Algebraic Topology to Group Theory

7.1. *Covering Spaces of a Graph*

Definition 7.1 (arc)**.** An arc is

$$\text{a topological space} \quad \text{homeomorphic to} \quad \text{the unit interval } [0,1].$$

If A is an arc, its endpoints p,q are the unique points of $\partial A = A \setminus \operatorname{int}(A)$ corresponding to $0,1$ under the homomorphism. Its interior is

$$\operatorname{int}(A) = A \setminus \partial A = A \setminus \{p,q\}.$$

Definition 7.2 (graph)**.** A graph (linear graph) is a topological space X satisfying the following properties: there exists a collection $\{A_\alpha\}_\alpha$ of subspaces of X, called the edges of X, such that the following hold:

- **(i)** for each α, the subspace A_α is an arc (so homeomorphic to $[0,1]$)
- **(ii)** for each $\alpha \neq \beta$, either $A_\alpha \cap A_\beta = \emptyset$ or $A_\alpha \cap A_\beta = \partial A_\alpha \cap \partial A_\beta$ which is a single point
- **(iii)** we have

$$X = \bigcup_\alpha A_\alpha \quad \text{and it is given with the colimit topology,}$$

i.e. a subset of X is open in X if and only if its intersection with each A_α is open in A_α

We say that

$$X^0 = \bigcup_\alpha \partial A_\alpha \quad \text{are the vertices of } X.$$

We say that

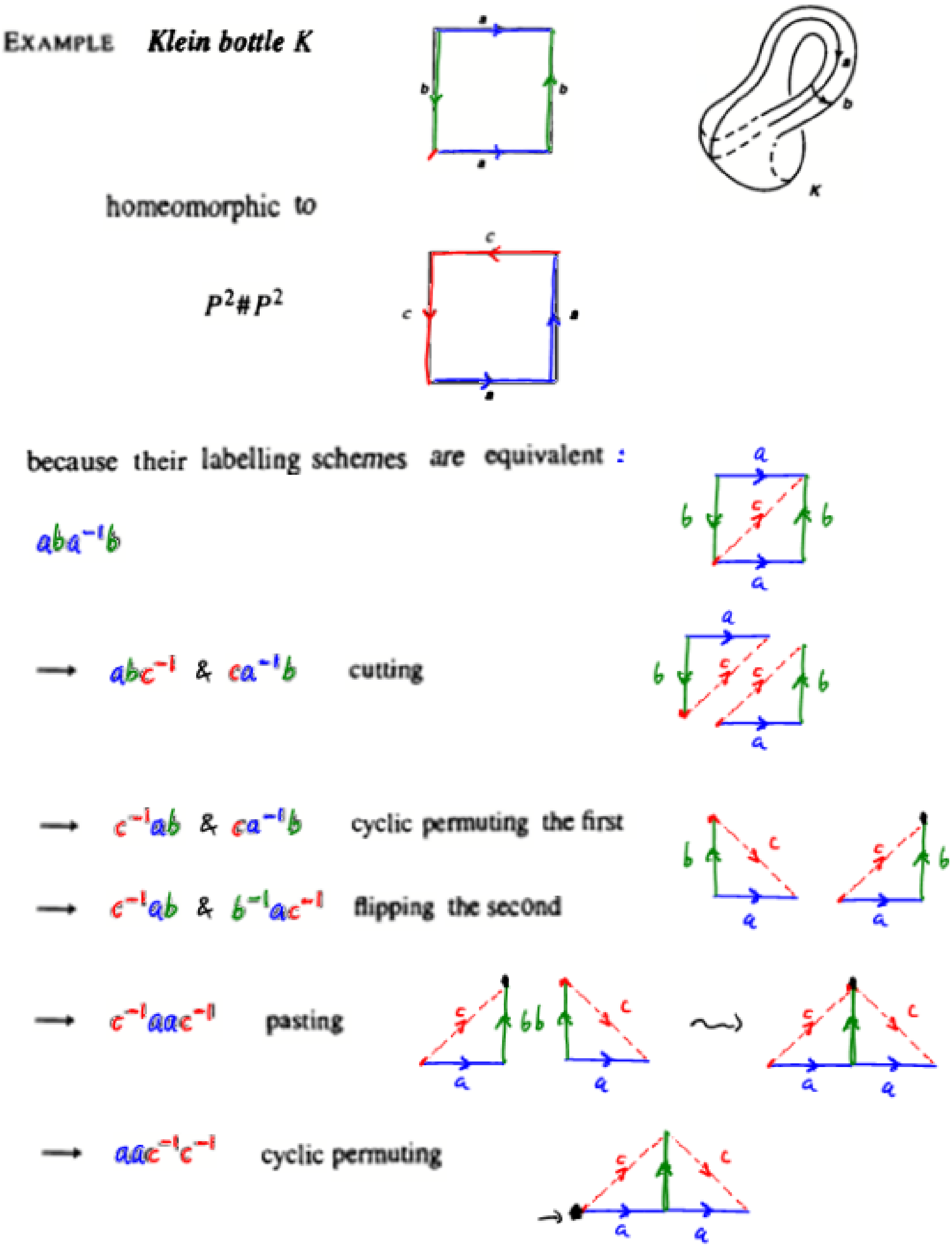
$$X \text{ is a finite graph} \quad \text{if and only if} \quad X^0 \text{ is finite.}$$

Example 7.1. Consider the diagrams in Figure 28. These are not graphs. However, they are homeomorphic to the graphs shown in Figure 29 respectively.

Example 7.2 (infinite earring/Hawaiian earring)**.** The infinite earring refers to the union of circles of radius $1/n$ in $\mathbb{R}^2$ with centre at the point $(1/n,0)$ given with the subspace topology of $\mathbb{R}^2$. It is

$$\text{not homeomorphic to a graph} \quad \text{as} \quad \text{it is not semilocally simply-connected.}$$

Definition 7.3 (wedge of circles)**.** A wedge of circles is a topological space X where there exists a collection $\{S_\alpha\}_\alpha$ of subspaces of X, called the circles of X, such that the following hold:



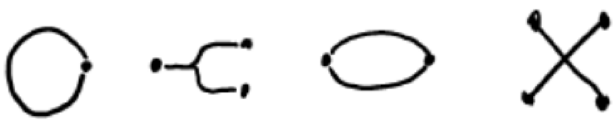


Figure 28: Non-examples of graphs

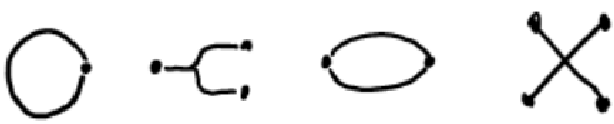


Figure 29: Diagrams in Figure 28 are homeomorphic to these respective graphs

(i) for each α, the subspace S_α is homeomorphic to the unit circle

(ii) there exists $p \in X$ such that for each $\alpha \neq \beta$, one has $S_\alpha \cap S_\beta = \{p\}$

(iii) we have

$$X = \bigcup_{\alpha} S_\alpha \quad \text{and it is given with the colimit topology.}$$

Example 7.3. A wedge of circles is homeomorphic to a graph. To see why, consider Figure 31, where we can write each subspace S_α as a graph having three edges, with p as one of its vertices.

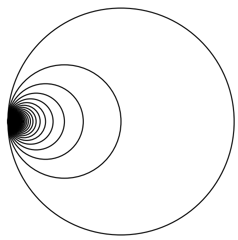


Figure 30: Hawaiian earring

Example 7.4. Let J be a discrete space, and let $E = [0,1] \times J$. The quotient space X obtained from E by collapsing $\{0\} \times J$ to a point p is a graph (Figure 32). As such, all the left endpoints of these segments (corresponding to 0 in each interval) are merged into one common point.

The graph obtained contains segments which can be thought of as rays that extend from the point p (the collapsed left endpoints) to the right, each reaching a different endpoint. This configuration resembles a star-shaped graph, with p as a central vertex.

Figure 31: Wedge of circles homeomorphic to a graph

Lemma 7.1. Let X be a graph. If

$$C \subseteq X \text{ is a union of edges and vertices} \quad \text{then} \quad C \text{ is closed in } X.$$

Proof. For each α, the intersection $C \cap A_\alpha$ is either empty, or A_α, or a subset of ∂A_α, so $C \cap A_\alpha$ is closed in A_α. So, C is closed in X. □

Corollary 7.1. Any subset of vertices is closed in X. Also, any union of edges is closed in X.

Lemma 7.2. Every graph X is normal, hence Hausdorff, as a topological space.

Definition 7.4 (subgraph)**.** Let X be a graph. A subgraph of X is a subset $Y \subseteq X$ which is a union of edges given with the subspace topology from X.

Lemma 7.3. If Y is a subgraph of X, then $Y \subseteq X$ is closed in X and the subspace topology of Y from X is the colimit topology from its edges. Hence, Y is itself a graph.

Lemma 7.4. Let X be a graph. For any compact subset $C \subseteq X$, there exists a finite subgraph $Y \subseteq X$ such that $C \subseteq Y$.

Lemma 7.5. Every graph is locally path-connected and locally simply-connected.

Theorem 7.1. Let X be a graph. Let $p : E \to X$ be a covering space of X. Then, E is a graph.

7.2. *The Fundamental Group of a Graph*

Definition 7.5 (oriented edge)**.** Let X be a graph. An edge e of X is oriented if and only if it is given with an ordering of ∂e from the initial vertex to the final vertex.

Definition 7.6 (edge path)**.** An edge path in X from x_0 to x_n is a finite sequence $x_0, x_1, \ldots, x_n$ of vertices in X such that for each $1 \leq i \leq n$, there exists an edge e_i of X with $\partial e_i = \{x_{i-1}, x_i\}$.

Definition 7.6 is equivalent to saying that an edge path is a finite sequence $e_1, \ldots, e_n$ of oriented edges of X such that for each $1 \leq i \leq n-1$,

$$\text{the final vertex of } e_i \quad \text{is} \quad \text{the initial vertex of } e_{i+1}.$$

Definition 7.7 (closed edge path)**.** An edge path in X is closed if and only if $x_0 = x_n$.

An edge path is not a reduced edge path if and only if there exists $1 \leq i \leq n-1$ such that the oriented edges e_i, e_{i+1} are the same edge, i.e. $\partial e_i = \partial e_{i+1}$. In this case, when we wish to reduce the edge path, we can

$$\text{delete } e_i \text{ and } e_{i+1} \quad \text{or equivalently} \quad \text{delte } x_{i-1} \text{ and } x_i$$

and the graph will still have an edge path (see Example 7.5).

Example 7.5 (reducing an edge path)**.** We can delete the edges e_4 and e_5 in order to reduce an edge path.

Given an oriented edge e, let

$$f_e : I \to e \hookrightarrow X \quad \text{be the composite path.}$$

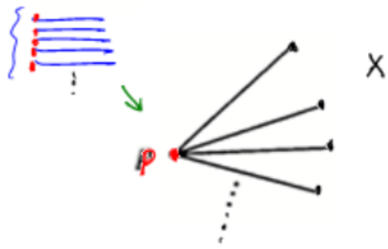


Figure 32: The quotient space X is a graph

Given an edge path $e_1,\dots,e_n$ from x_0 to x_n, the corresponding path from x_0 to x_n is

$$f = f_1 * (f_2 * (\dots * f_n)) \quad \text{where} f_i = f_{e_i}.$$

It is a loop if and only if the edge path is closed.

Lemma 7.6. A graph X is connected as a topological space if and only if

for every pair of vertices $x, y \in X$ there exists a reduced edge path in X from x to y.

Definition 7.8 (tree)**.** A tree is a graph which

is connected and contains no closed reduced edge paths.

Definition 7.9 (subtree)**.** A subtree of a tree in a graph X is a subgraph of X which is a tree.

Example 7.6. Here are some examples of trees.

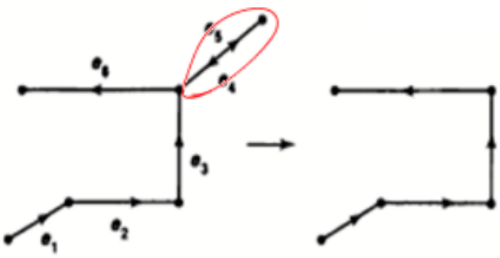


Figure 33: Reducing an edge path

Example 7.7 (infinite trees)**.** Here are more examples of trees which have an infinite number of vertices. THe right tree is quite interesting (called a complete binary tree).

Example 7.8. Here is a non-example of a tree.

Lemma 7.7. Let X be a graph, and let T be a tree in X. For any edge A of X such that $T \cap A$ is a single vertex, the union $T \cup A$ is a tree in X.

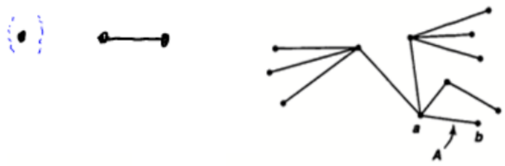


Figure 34: Examples of trees

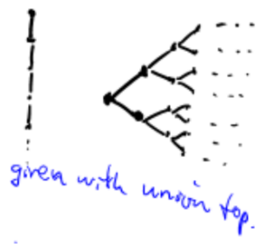


Figure 35: More examples of trees

If T is finite and consists of at least one edge, then there exists a tree T_0 in X and an edge A of X such that $T_0 \cap A$ is a single vertex and $T = T_0 \cup A$.

Theorem 7.2. A tree is simply-connected as a topological space.

Definition 7.10 (maximal tree)**.** Let X be a graph. A maximal tree in X is a tree T in X such that there does not exist a tree in X that properly contains T.

Theorem 7.3. Let X be a connected graph. A tree T in X is maximal if and only if

$$X^0 \subseteq T \quad \text{or equivalently} \quad T \text{ contains all the vertices of } X.$$

Theorem 7.4. Any tree T_0 in X is contained in a maximal tree in X.

Theorem 7.5 (fundamental group of a graph)**.** Let X be a connected graph, $T \subseteq X$ be a maximal tree in X, and $J \subseteq X$ be the set of edges of X not in T. Then, the fundamental group of X is a free group on the set J, i.e.

$$\text{for any vertex } x_0 \text{ of } T \quad \text{we have} \quad \pi_1(X, x_0) = \{ (J).$$